Instructions: Attempt any **five** questions, and please provide **careful and complete answers with proofs**. If you attempt more questions, **specify** which five should be graded.

1. Let x and y be elements of a group G such that x has order 3, and y is not the identity and has odd order. If $xyx^{-1} = y^{17}$, determine the order of y.

2. Let G be a finite group and p the smallest prime dividing the order of G. Let H be a subgroup of index p. Prove that H is normal in G.

3. Does there exist a field K such that the multiplicative group $K^* = K \setminus \{0\}$ is isomorphic to the Klein-4-group, $\mathbb{Z}_5 \times \mathbb{Z}_7$?

4. Consider the field $K = \mathbb{Q}(\sqrt{2} + \sqrt{7})$. Find the set $\mu(K)$ of all roots of unity K and describe which abelian group it is isomorphic to.

5. Is every ideal of the ring $\mathbb{Z} \times \mathbb{Z}$ a principal ideal? Prove or disprove.

6. Let R denote the set of all periodic sequences of real numbers, ie:

$$R = \{\alpha : \mathbb{N} \to \mathbb{R}, \alpha_{n+d} = \alpha_n \text{ for some } d\}.$$

R is a ring with point-wise addition and multiplication:

$$(\alpha + \beta)_n = \alpha_n + \beta_n, \quad (\alpha\beta)_n = a_n\beta_n.$$

(a) Is R an integral domain? Justify your answer.
(b) Let $\mathbf{c}$ denote the constant sequence $(c, c, c, \dots)$ for $c \in \mathbb{R}$. Find all solutions in R of the polynomial equation:

$$x^2 - \mathbf{1} = \mathbf{0}$$

7. Let A be a square matrix with complex entries, and ϵ a positive real number. Prove that $\epsilon I + A^* A$ is nonsingular.

8. Consider a group homomorphism $f : \mathbb{Z}^4 \to \mathbb{Z}^2$. List, up to isomorphism, all the possibilities for the kernel of f. Hint: It is a finite list.