Instructions: Attempt any **five** questions, and please provide **careful and complete answers with proofs**. If you attempt more questions, specify which five should be graded. Otherwise, by default, only the first five will be graded.

1. **(a)** For $1 < p < \infty$, show that for each $f \in L^p([0,1], \mathrm{d}x)$ there is a *unique* $g \in L^q([0,1], \mathrm{d}x)$, where $1/p + 1/q = 1$ so that

$$\int_{[0,1]} fg\,\mathrm{d}x = \|f\|_p \qquad \text{and} \qquad \|g\|_q = 1 \ . \tag{$*$}$$

(b) Give an example of an $f \in L^1([0,1], \mathrm{d}x)$ for which there are *infinitely many* $g \in L^\infty([0,1], \mathrm{d}x)$ so that $(*)$ holds.

(c) Give an example of an $f \in L^\infty([0,1], \mathrm{d}x)$ for which there is *no* $g \in L^1([0,1], \mathrm{d}x)$ so that $(*)$ holds.

2. Is there a function $f \in L^p([0,1], \mathrm{d}x)$ for all $1 \le p < \infty$ such that for each x in $[0,1]$,

$$\limsup_{z \to x} f(z) = +\infty \qquad \text{and} \qquad \liminf_{z \to x} f(z) = -\infty \ ?$$

Either prove that there is no such function, or give an example.

3. **(a)** Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $1 < p < \infty$, and suppose that f is a measurable function on X such that for some $C < \infty$

$$\int_A |f(x)| \mathrm{d}\mu \leq C \mu(A)^{1/p'} \tag{$*$}$$

for every measurable set $A \subset X$, where $1/p + 1/p' = 1$. Show that this does not imply that $f \in L^p(X, \mathcal{S}, \mu)$.

(b) Suppose in addition to $(*)$ that for some q with $p < q < \infty$, there is a constant $D < \infty$ such that

$$\int_A |f(x)| \mathrm{d}\mu \leq D \mu(A)^{1/q'} \tag{$**$}$$

for every measurable set $A \subset X$, where $1/q + 1/q' = 1$. Show that then $f \in L^r(X, \mathcal{S}, \mu)$ for each r with $p < r < q$.

4 Let $F(x,y)$ be a continuous function on $[0,1] \times [0,1]$. Define a linear transformation $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ by

$$Tf(x) = \int_0^1 F(x,y)f(y)\mathrm{d}y \ .$$

Show that if $\{f_n\}$ is any sequence in $\mathcal{C}([0,1])$ with

$$\sup_n \|f_n\|_{\mathcal{C}([0,1])} < \infty \ ,$$

then there is a subsequence of $\{Tf_n\}$ that is strongly convergent in $\mathcal{C}([0,1])$.

5 Let S be a closed linear subspace of $L^1([0,1])$ with the property that for each individual $f \in S$, there is some $p > 1$ so that $f \in L^p([0,1])$. Show that there is then some $p > 1$ so that $S \subset L^p([0,1])$.

6. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f \in L^1(X, \mu)$. Show that there exists a convex increasing function $\phi : [0, \infty) \to \mathbf{R}$ such that

$$\phi(0) = 0, \quad \lim_{t \to \infty} \frac{\phi(t)}{t} = \infty,$$

and

$$\phi(|f|) \in L^1(X, \mu).$$

7. Let $f : [0, 1] \to \mathbf{R}$ be continuous, $g : [0, 1] \to \mathbf{R}$ Lebesgue measurable, and $0 \leq g(x) \leq 1$ for a.e. $x \in [0, 1]$. Find the limit:

$$\lim_{n\to\infty} \int_0^1 f\left(g(x)^n\right) dx.$$

8. Let X and Y be metric spaces and $f : X \to Y$ be a mapping. Show that if the restriction of f on any compact subset of X is continuous, then f is continuous on X.