Instructions: Attempt any **five** questions, and please provide **careful and complete answers with proofs**. If you attempt more questions, specify which five should be graded. Otherwise, by default, only the first five will be graded.

1. (a) For $1 < p < \infty$, show that for each $f \in L^p([0,1], \mathrm{d}x)$ there is a *unique* $g \in L^q([0,1], \mathrm{d}x)$, where $1/p + 1/q = 1$ so that

$$\int_{[0,1]} fg\mathrm{d}x = \|f\|_p \qquad \text{and} \qquad \|g\|_q = 1 \ . \tag{$*$}$$

(b) Give an example of an $f \in L^1([0,1], \mathrm{d}x)$ for which there are *infinitely many* $g \in L^\infty([0,1], \mathrm{d}x)$ so that $(*)$ holds.

(c) Give an example of an $f \in L^\infty([0,1], \mathrm{d}x)$ for which there is *no* $g \in L^1([0,1], \mathrm{d}x)$ so that $(*)$ holds.

SOLUTION By Hölders inequality, and then the hypothesis that $\|g\|_q = 1$,

$$\left| \int_{[0,1]} fg\mathrm{d}x \right| \le \|f\|_p \|g\|_q = \|f\|_p \ .$$

Hence for $(*)$ to hold, g must be such that there is equality in Hölder's inequality. For $p < \infty$, this is the case if and only if g is a constant multiple of $|f(x)|^{p-2}f^*(x)$. (Those who do not remember the exact condition for the cases of equality can easily derive them if they remember the proof in terms of the arithmetic–geometric mean inequality.)

It is now easy to answer the questions.

(a) For $1 < p < \infty$, let

$$g(x) = \frac{|f(x)|^{p-2}f^*(x)}{\|f\|_p^{p-1}} \ . \tag{$**$}$$

By what we have noted above concerning case of equality in Hölder's inequality, if there is any such g, this must be it Let's check that it works.

Since $q = p/(p-1)$, $|g(x)|^q = |f(x)|^p/\|f\|_p^p$ so that $\int_{[0,1]} |g(x)|^q \mathrm{d}x = \|f\|_p^p/\|f\|_p^p = 1$. That, $\|g\|_q = 1$. Also

$$\int_{[0,1]} fg\mathrm{d}x = \int_{[0,1]} |f|^p/\|f\|_p^{p-1}\mathrm{d}x = \|f\|_p \ .$$

So this works and hence $(**)$ gives the unique element of $L^q([0,1], \mathrm{d}x)$ for which $(*)$ holds.

(b) Suppose that $f(x) = 2$ on $[0, 1/2]$ and $f(x) = 0$ elsewhere. For any number a with $|a| \le 1$, let $g(x) = 2$ on $[0, 1/2]$ and $g(x) = a$ elsewhere.

Then $\int_{[0,1]} fg\mathrm{d}x = 1 = \|f\|_1$ while $\|g\|_\infty = 1$. Since there are infinitely many a with $|a| \le 1$, there are infinitely many such functions g.

(c) Let $f(x) = x$. Then $\|f\|_\infty = 1$. Let g be any function in $L^1([0,1])$ with $\|g\|_1 = 1$. Then by dominated convergence, there is some n so that

$$\int_{[0,1-1/n]} |g(x)|\mathrm{d}x \geq 1/2 \ .$$

But then

$$\begin{aligned}
\left|\int_{[0,1]} fg\mathrm{d}x\right| &\leq \int_{[0,1-1/n]} |fg|\mathrm{d}x + \int_{[1-1/n,1]} |fg|\mathrm{d}x \\
&\leq (1-1/n)\int_{[0,1-1/n]} |g|\mathrm{d}x + \int_{[1-1/n,1]} |g|\mathrm{d}x \\
&= |\int_{[0,1]} |g|\mathrm{d}x - \frac{1}{n}\int_{[0,1-1/n]} |g|\mathrm{d}x \\
&= |\int_{[0,1]} |g|\mathrm{d}x - \frac{1}{n}\int_{[0,1-1/n]} |g|\mathrm{d}x \\
&\leq 1 - 1/2n < 1 \ .
\end{aligned}$$

So there is no such g for this f.

2. Is there a function $f \in L^p([0,1], \mathrm{d}x)$ for all $1 \le p < \infty$ such that for each x in $[0,1]$,

$$\limsup_{z \to x} f(z) = +\infty \qquad \text{and} \qquad \liminf_{z \to x} f(z) = -\infty \ ?$$

Either prove that there is no such function, or give an example.

SOLUTION There are such functions. For $x \in R$, let $\phi(x) = \ln(|x|) for |x| \le 1$, and 0 otherwise. This is in L^p for each $p < \infty$, by comparison with a small negative power of $|x|$, but $\phi(0) = \infty$. Next, for any numbers a and b, let $\phi_{a,b}(x) = \phi(a(x-b))$. Then

$$\|phi_{a,b}\|_p = |a|^{-1/p} \|\phi\|_p \ ,$$

and $\phi_{a,b}(b) = \infty$. Now let $\{q_n\}$ be some enumeration of the rational numbers in $[0,1]$. Define

$$f(x) = \sum_{n=1}^{\infty} \phi_{2^n, q-n}(x) \ ,$$

restricted to $[0,1]$. Then by Minkowskii's inequality,

$$\|f\|_p \le \sum_{n=1}^{\infty} 2^{-n/p} \|\phi\|_p$$

which is finite for all $p < \infty$. Thus, $f \in L^p([0,1], \mathrm{d}x)$ for all $p < \infty$. Also clearly at each rational number q, $f(q) = \infty$, and since each interval around each x in $[0,1]$ contains rational numbers,

$$\limsup_{z \to x} f(z) = +\infty \qquad \text{and} \qquad \liminf_{z \to x} f(z) = -\infty \ .$$

3. **(a)** Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $1 < p < \infty$, and suppose that f is a measurable function on X such that for some $C < \infty$

$$\int_A |f(x)|\mathrm{d}\mu \le C\mu(A)^{1/p'} \tag{$*$}$$

for every measurable set $A \subset X$, where $1/p + 1/p' = 1$. Show that this does not imply that $f \in L^p(X, \mathcal{S}, \mu)$.

(b) Suppose in addition to $(*)$ that for some q with $p < q < \infty$, there is a constant $D < \infty$ such that

$$\int_A |f(x)|\mathrm{d}\mu \le D\mu(A)^{1/q'} \tag{$**$}$$

for every measurable set $A \subset X$, where $1/q + 1/q' = 1$. Show that then $f \in L^r(X, \mathcal{S}, \mu)$ for each r with $p < r < q$.

SOLUTION (a) Let $X = R_+$, and let μ be Lebesgue measure. Let $f(x) = 1/x^{1/p}$. Then if $\mu(A) = a$, it is clear that

$$\int_A |f(x)|\mathrm{d}\mu \le \int_{[0,a]} |f(x)|\mathrm{d}\mu = \int_{[0,a]} x^{-1/p}\mathrm{d}x = \frac{1}{p'}a^{-1/p'} \ .$$

Thus, $(*)$ holds with $C = 1/p'$. However, f is not in L^p.

(b) For each $t > 0$, let $h(t) = \mu(\{x \ : \ |f(x)| > t \ \})$. Then for any $1 < r < \infty$,

$$\|f\|_r^r = -\int_0^\infty t^p \mathrm{d}h(t) = r\int_0^\infty t^{r-1}h(t)\mathrm{d}t \ . \tag{$**$}$$

We can use $(**)$ to estimate L^r norms if we can estimate $h(t)$. We can do this using $(*)$ and $(**)$ if we consider the set $A = \{x \ : \ |f(x)| > t \ \}$. By $(*)$,

$$th(t) \le \int_{\{x \ : \ |f(x)|>t \ \}} |f(x)|\mathrm{d}\mu \le Ch(t)^{1-1/p} \ .$$

That is,

$$h(t) \le (C/t)^p \ .$$

Likewise, form $(**)$ we deduce

$$h(t) \le (C/t)^q \ .$$

We then have

$$\begin{aligned} r\int_0^\infty t^{r-1}h(t)\mathrm{d}t &\le r\int_0^1 t^{r-1}(C/t)^p\mathrm{d}t + r\int_1^\infty t^{r-1}(D/t)^r\mathrm{d}t \\ &\le rC^p\int_0^1 t^{r-1-p}\mathrm{d}t + \le rD^q\int_1^\infty t^{r-1-q}\mathrm{d}t \ . \end{aligned}$$

Since $p < r < q$, both of these integral converge, and then by $(**)$, $f \in L^r$.

4 Let $F(x,y)$ be a continuous function on $[0,1] \times [0,1]$. Define a linear transformation $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ by

$$Tf(x) = \int_0^1 F(x,y)f(y)\mathrm{d}y \ .$$

Show that if $\{f_n\}$ is any sequence in $\mathcal{C}([0,1])$ with

$$\sup_n \|f_n\|_{\mathcal{C}([0,1])} < \infty \ ,$$

then there is a subsequence of $\{Tf_n\}$ that is strongly convergent in $\mathcal{C}([0,1])$.

SOLUTION Use Arzelà–Ascoli to get compactness...

5 Let S be a closed linear subspace of $L^1([0,1])$ with the property that for each individual $f \in S$, there is some $p > 1$ so that $f \in L^p([0,1])$. Show that there is then some $p > 1$ so that $S \subset L^p([0,1])$.

SOLUTION Let $S_n = S \cap L^{1+1/n}([0,1])$. Show that these sets are closed in S using Fatou, pointwise convergent subsequence... Then apply Baire's Theorem.

6. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f \in L^1(X, \mu)$. Show that there exists a convex increasing function $\phi : [0, \infty) \to \mathbf{R}$ such that

$$\phi(0) = 0, \quad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty,$$

and

$$\phi(|f|) \in L^1(X, \mu).$$

SOLUTION Recall that for any measurable function f on $(X, \mathcal{S}, \mu)$,

$$\int_X |f| d\mu = \int_0^\infty \omega(\lambda) d\lambda,$$

where

$$\omega(\lambda) = \mu\left(\{x \in X : |f(x)| > \lambda\}\right) \qquad \text{for } \lambda \in [0, \infty).$$

This can be proved using Fubini's theorem. In particular, $f \in L^1(X, \mu)$ if and only if $\omega(\lambda)$ is in $L^1[0, \infty)$.

We first consider the trivial case where $|f| \in L^\infty(X, \mu)$. Denote $M = \|f\|_{L^\infty} < \infty$. Define

$$\phi(t) = t \quad (0 \le t \le M), \qquad \phi(t) = t + (t - M)^2 \quad (t > M).$$

Then ϕ satisfies the requirements.

Next consider the case where $|f| \notin L^\infty(X, \mu)$. Let

$$\rho(\lambda) = \int_\lambda^\infty \omega(t) dt \qquad t \ge 0.$$

We see that $\rho(0) = \|f\|_{L^1} < \infty$, $\rho(\lambda)$ is positive and decreasing, $\rho(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$, $\rho(\lambda)$ is absolutely continuous, and $\rho'(\lambda) = -\omega(\lambda)$ a.e. $\lambda \in (0, \infty)$. Define

$$\phi(0) = 0, \qquad \phi(t) = \int_0^t \rho(\lambda)^{-1/2} d\lambda \qquad t \ge 0.$$

Since $\phi'(t) = \rho(t)^{-1/2} > 0$ is increasing, it follows that $\phi(t)$ is convex and strictly increasing. It is also easy to see

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \rho(\lambda)^{-1/2} d\lambda = \infty,$$

since $\rho(\lambda)^{-1/2} \to \infty$ as $\lambda \to \infty$. Finally, let us verify $\phi(|f|) \in L^1(X, \mu)$. Notice that

$$\mu\left(\{x \in X : \phi(|f(x)|) > \lambda\}\right) = \mu\left(\{x \in X : |f(x)| > \phi^{-1}(\lambda)\}\right) = \omega(\phi^{-1}(\lambda)).$$

Hence,

$$
\begin{aligned}
\int_X \phi(|f|)d\mu &= \int_0^\infty \omega(\phi^{-1}(\lambda))d\lambda \\
&= \int_0^\infty \omega(t)\phi'(t)dt \qquad \text{(change of variable } \lambda = \phi(t)\text{)} \\
&= \int_0^\infty \omega(t)\rho(t)^{-1/2}dt \\
&= \int_0^\infty -\rho'(t)\rho(t)^{-1/2}dt \\
&= \Big[-2\rho(t)^{1/2}\Big]_0^\infty \qquad \text{(since } \rho(t) \text{ is abs. conti.)} \\
&= 2\rho(0)^{1/2} = 2\|f\|_{L^1}^{1/2} < \infty.
\end{aligned}
$$

7. Let $f : [0,1] \to \mathbf{R}$ be continuous, $g : [0,1] \to \mathbf{R}$ Lebesgue measurable, and $0 \le g(x) \le 1$ for a.e. $x \in [0,1]$. Find the limit:

$$\lim_{n\to\infty} \int_0^1 f\left(g(x)^n\right) dx.$$

SOLUTION Define

$$A = \{x \in [0,1] : g(x) = 1\}, \quad B = \{x \in [0,1] : 0 \le g(x) < 1\}.$$

By the assumption, $A \cup B$ is of full measure in $[0,1]$; that is, $\mu(A) + \mu(B) = 1$.

For every $x \in A$, $f\left(g(x)^n\right) = 1$.

For every $x \in B$, $g(x)^n \to 0$ as $n \to \infty$. Combining this with the continuity of f, we obtain $f\left(g(x)^n\right) \to f(0)$.

Since f is continuous on a compact set $[0,1]$, $|f|$ is bounded: $|f(t)| \le M < \infty$ for all $x \in [0,1]$. This implies the boundedness of the integrand:

$$|f\left(g(x)^n\right)| \le M \qquad \text{for all } x \in A \cup B.$$

By Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^1 f\left(g(x)^n\right) dx = \int_A f(1) dx + \int_B f(0) dx = f(1)\mu(A) + f(0)[1 - \mu(A)].$$

8. Let X and Y be metric spaces and $f : X \to Y$ be a mapping. Show that if the restriction of f on any compact subset of X is continuous, then f is continuous on X.

SOLUTION Let $x_n \to x$ in X. We need to show $f(x_n) \to f(x)$. Define

$$K = \{x_1, x_2, \cdots\} \cup \{x\}.$$

It is easily seen that K is a compact subset of X. By the assumption, f is continuous on K. This implies $f(x_n) \to f(x)$.