Comprehensive Examination, February 2003

REAL ANALYSIS

Instructions: (1) Please do any 5 of the 7 problems. (2) Be sure to justify your assertions. Please provide careful and complete answers; partial progress towards many questions counts less than a complete answer to fewer questions. If you attempt more than five questions, specify which ones you want to be graded, otherwise the first five answered will be graded.

1. Proof or counterexample: If f is a nonnegative function in $L^1[0,1]$ and $\int_0^1 f(x)dx = 1$, then there exists a measurable set $A \subset [0,1]$ such that

$$\mu(A) = \frac{1}{2} \qquad \text{and} \qquad \int_A f = \frac{1}{2}$$

(μ denotes Lebesgue measure).

2. Let μ denote Lebesgue measure on $[0,1]$, ν counting measure on $[0,1]$, and $\mathcal{H}$ Hausdorff measure on $[0,1]$. Hausdorff measure is given by

$$\mathcal{H}(A) = \lim_{r \searrow 0} \inf \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j, \operatorname{diam}(A_j) \le r \right\}$$

on

$$\mathcal{M} = \{A \subset [0,1] : \mathcal{H}(A \cap B) + \mathcal{H}(B \backslash A) = \mathcal{H}(B) \text{ for every } B \subset [0,1]\}.$$

Here, $\operatorname{diam}(A_j) = \sup\{|x-y| : x, y \in A_j\}$ is the diameter of A_j.

(a) Give precise definitions for μ and ν.

(b) Determine which of the following assertions concerning absolute continuity are true and which are false on the intersection of the domains of the given measures:

$$\mu \ll \nu, \quad \mu \ll \mathcal{H}, \quad \nu \ll \mu, \quad \nu \ll \mathcal{H}, \quad \mathcal{H} \ll \mu, \quad \mathcal{H} \ll \nu.$$

(Justify your answers.)

(c) Is there a function f such that $\mu(A) = \int_A f \, d\nu$ for every $A \in \mathcal{M}$? (Justify your answer.)

3. Suppose $f_n, f \in L^1(-\infty,\infty)$, $\|f_n\|_{L^1(-\infty,\infty)} \leq 1$ for all n, and $\displaystyle\int_E f_n \to \int_E f$ for every measurable set E. Prove that if g is a measurable function with $0 \leq g \leq 1$ a.e., then $\displaystyle\int f_n g \to \int fg$.

4.(a) Give a counterexample to the statement: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue measurable functions on $(-\infty,\infty)$ satisfying $f_1 > f_2 > \cdots > 0$ a.e. and $f_n \to f$ a.e., then

$$\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f.$$

(b) Add one additional hypothesis to the statement of part (a) (do not otherwise modify the given hypotheses) and prove the resulting assertion.

5. Denote the class of absolutely summable sequences of real numbers by l^1. Let $\alpha = \{\alpha_n\}_{n=1}^{\infty}$, $\beta = \{\beta_n\}_{n=1}^{\infty} \in l^1$ with $\alpha_n, \beta_n > 0$ for all n. Assume $\|\beta\| = 1$. Show that

$$\prod_{n=1}^{\infty} \alpha_n^{\beta_n} \leq \sum_{n=1}^{\infty} \alpha_n \beta_n < \infty.$$

6. State carefully the Riesz Representation Theorem for linear functionals on L^p, $1 \leq p < \infty$, and prove the uniqueness of Riesz representation.

7. Let $A = \big\{ f \in L^1[0,1] : |f(x)| \geq 1 \text{ a.e.} \big\}$ True or False

(a) A is norm closed in $L^1[0,1]$.

(b) A is weakly closed in $L^1[0,1]$.

Comprehensive Examination, Spring 2003

ALGEBRA

Instructions: (1) Please do any 5 of the 7 problems. (2) Be sure to justify your assertions. Please provide careful and complete answers; partial progress towards many questions counts less than a complete answer to fewer questions. If you attempt more than five questions, specify which ones you want to be graded, otherwise the first five answered will be graded.

1. Let G be a finite group. For any $x \in G$, let $Z(x) = \{g \in G : gx = xg\}$. Let $\mathcal{C}(G) = \{Z(x) : x \in G\}$. Prove the following statements.
   (a) If $|\mathcal{C}(G)| = 1$ then G is Abelian.
   (b) $|\mathcal{C}(G)| \neq 2$.
   (c) $|\mathcal{C}(G)| \neq 3$.

2. (a) Show that if H and K are normal subgroups of a group and $H \cap K = \{1\}$ where 1 is the identity, then $xy = yx$ for all $x \in H$ and $y \in K$.
   (b) Let G be a group of order pq, where $p < q$ and both p and q are prime numbers. Let P be a subgroup of G of order p and Q a subgroup of G of order q. Prove that Q is a normal subgroup of G, and if P is a normal subgroup of G then G is cyclic.

3. Let R be an integral domain, and let $R\{x\}$ denote the set of formal power series in x with coefficients in R. Then $R\{x\}$ is a commutative ring under the following operations:
$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n, \qquad \text{and}$$
$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n.$$
   Prove the following statements.
   (a) $I = (x)$, the principle ideal generated by x, is a prime ideal in $R\{x\}$.
   (b) I is a maximal ideal if and only if R is a field.

4. Let F be a finite field. Prove (from first principles) that there exist a prime number p and a positive integer n such that $|F| = p^n$.

5. Let $\mathbb{Z}_p[x]$ denote the polynomial ring with coefficients in $\mathbb{Z}_p$ (where p is a prime number) and let $f(x)$ be an irreducible polynomial over $\mathbb{Z}_p$ of degree $n > 0$. Show (from first principles) that $\mathbb{Z}_p[x]/(f(x))$ is a field with p^n elements. Here, $f(x))$ is the ideal in $\mathbb{Z}_p[x]$ generated by $f(x)$.

6. Prove that $\langle A, B\rangle = \text{trace}(AB^T)$ defines an inner product on the space of $n \times n$ real matrices, and find the orthogonal complement of the subspace of all skew symmetric matrices.

7. Prove that diagonalizable matrices A and B can be simultaneously diagonalized (there exists a matrix S with $S^{-1}AS$ and $S^{-1}BS$ both diagonal) if and only if $AB = BA$.