Algebra, Spring 2004.

Attempt any five questions, and please provide careful and complete answers. If you attempt more than five questions, specify which should be graded.

1. How many elements of the permutation group S_7 commute with the permutation $(12)(34) \in S_7$?

2. Recall that $SL_2(\mathbb{Z}/5\mathbb{Z})$ is the group of 2×2 matrices of determinant 1, which have entries in $\mathbb{Z}/5\mathbb{Z}$. Determine the number of 5-Sylow subgroups of $SL_2(\mathbb{Z}/5\mathbb{Z})$.

3. Let I be the ideal of the polynomial ring $\mathbb{R}[x]$ generated by the elements $x^3 + 1$ and $x^4 + 3x + 2$. Is it possible to generate I by one element? Is I a prime ideal? Is it maximal?

4. Let K be a field with 49 elements. For each of the polynomials below, determine the number of distinct roots in K.
$$x^{48} - 1, \qquad x^{49} - 1, \qquad x^{54} - 1.$$

5. Let $\xi = e^{2\pi i/5}$. Determine the degrees of the field extensions
   (a) $[\mathbb{Q}(\xi) : \mathbb{Q}]$,
   (b) $[\mathbb{Q}(\xi, \sqrt{5}) : \mathbb{Q}]$.

6. Determine all 3×3 matrices M with rational entries for which M^7 is the identity matrix.

7. (a) Let A and B be $n \times n$ matrices over the complex numbers. Recall that A^* denotes the conjugate of the transpose of A. If $A^*A + B^*B = 0$, prove that $A = B = 0$.

   (b) Let A and B be $n \times n$ positive definite Hermitian matrices over the complex numbers. Is it true that the eigenvalues of AB are strictly positive? Prove, or give a counterexample.