FUNCTIONAL ANALYSIS Spring 2003

SOLUTIONS TO SOME PROBLEMS

Warning:These solutions may contain errors!!

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PROBLEM 1. Prove that a necessary and sufficient condition that the metric space $\big(X, d\big)$ be complete is that every nested sequence of nonempty closed sets $(F_i)_{i=1}^{\infty}$ with diameters tending to 0 , has a nonempty intersection:

$$\bigcap_{i=1}^{\infty} F_i \neq \emptyset.$$

SOLUTION.
($\Longrightarrow$:) Suppose first that $\big(X, d\big)$ is complete. We shall show that the above condition is satisfied under this assumption. Suppose that (F_i) (i=1,2,...) is a nested sequence of nonempty closed sets such that $diam(F_i) := \sup_{a,b,\in F_i} d(a,b) \to 0$. By selecting a point $x_n \in F_n$ for each $n = 1, 2, ..$, we can generate a sequence (x_n). This sequence must be Cauchy sequence, because, assuming $m > n$, we have $d(x_m, x_n) \leq diam(F_n)$, which tends to). Since $\big(X, d\big)$ is complete, (x_n) must have a limit x. Now, for any given m, we have a sequence of points $\{x_m, x_{m+1}, ...\} \subset F_n$. By this we have, $x \in closure(F_n) = F_n, \forall n$. Hence, $x \in \bigcap_{i=1}^{\infty} F_i$.
(:$\Longleftarrow$) Conversely, suppose that the given condition is satisfied and let (x_n) be a Cauchy sequence. Letting $H_n = \{x_n, x_{n+1}, ...\}$, we can say that, since (x_n) is a Cauchy sequence, $diam(H_n) \to 0$. It is also true that $diam(closure(H_n)) \to 0$. Further, since

$$H_{n+1} \subset H_n \Rightarrow closure(H_{n+1}) \subset closure(H_n),$$

we see that $(closure(H_n))$ is a closed, nested sequence of nonempty sets in X, whose diameters tend to 0. By the hypothesis, we can conclude the existence of an x such that

$$x \in \bigcap_{i=1}^{\infty} closure(H_n).$$

Therefore, since $d(x_n, x) \leq diam(closure(H_n)) \to 0$, we see that $x_n \to x$.

PROBLEM 2. Let f be acontinuous function such that $f : X \to Y$ where X and Y are metric spaces. Suppose A is a compact subset of X, prove that $f(A)$ is a compact subset of Y.

SOLUTION. Let (G_α) be a collection of open sets in Y such that

$$f(A) \subset \bigcup_\alpha G_\alpha.$$

Now we have,

$$A \subset f^{-1}(f(A)) \subset f^{-1}\big(\bigcup_\alpha G_\alpha\big) = \bigcup_\alpha f^{-1}(G_\alpha).$$

Since f is continuous each of the $f^{-1}(G_\alpha)$ are open sets. Thus, the family $(f^{-1}(G_\alpha)$ is an open covering for A. But by assumption A is compact, so there exist $G_1, G_2, ..., G_n$ such that

$$A \subset \bigcup_{i=1^n} f^{-1}(G_i),$$

which implies

$$f(A) \subset \bigcup_{i=1}^{n} f\big(f^{-1}(G_i)\big) \subset \bigcup_{i=1}^{n} G_i.$$

We have succeeded in selecting a finite subcovering from the original open covering for $f(A)$, and that shows $f(A)$ is compact.

PROBLEM 3. We know that if (X, d) is a metric space. Then $A \subset X$ is compact only if A is closed and bounded. In R^n we know that the converse is also true which is precisely the statement of the Heine-Borel Theorem. Give a counterexample to show that the converse is not necessarily true if we have an arbitrary metric space.

SOLUTION. Suppose X is any infinite set and the trivial metric d is assigned to it. For $\epsilon < \frac{1}{2}$ we have

$$S_\epsilon(x) = \{x\} \subset \{x\}$$

from which we conclude that even one-point sets in this metric space are open sets. X is itself bounded set as the distance between any two points is at most 1 and even this space is complete(exercise!). Since X is the whole space X is also closed. Now if we consider the following open covering of X,

$$X \subset \bigcup_{x \in X} \{x\}.$$

It is clear that no finite subcovering can be chosen from this covering, so X is not compact though it is closed and bounded.

PROBLEM 4. For each n let $f_n : R \to R$ be a differentiable function. Suppose also that for each n and x we have $|f_n'| \leq 1$. Show that if for all x $\lim_{n\to\infty} f_n(x) = g(x)$ then $g : R \to R$ is a continuous function.

SOLUTION. Fix $\epsilon > 0$. For real numbers $a < b$, we can choose n so large that $|f_n(a)-g(a)| < \epsilon$ and $|f_n(b)-g(b)| < \epsilon$. Then by the Mean Value Theorem, for $a < \xi < b$, we have the following estimate:

$$|g(a) - g(b)| \leq |g(a) - f_n(a)| + |f_n(a) - f_n(b)| + |f_n(b) - g(b)| < 2\epsilon + |f_n'(\xi)||b - a|.$$

Since $\epsilon > 0$ is arbitrary, the above estimate implies, $|g(a)-g(b)| \leq |b-a|$ and this shows that g is continuous.

PROBLEM 5. Let $f_n : [0,1] \to [0,\infty)$ be continuous for $n = 1, 2, ...$ Suppose

$$(*) f_1(x) \geq f_2(x) \geq f_3 \geq ...$$

for $x \in [0,1]$ and let $f(x) = \lim_{n\to\infty} f_n(x), M = \sup_{x\in[0,1]} f(x)$.
a.)Show that there exists $t \in [0,1]$ such that $f(t) = M$.
b.)Does the conclusion remain valid if we replace the condition $(*)$ by the follwing condition? *Suppose there exists* n_x *such that* $f_n(x) \geq f_{n+1}(x)$ *for all* $x \in [0,1]$ *and* $n > n_x$.

SOLUTION.
a.) For $\epsilon > 0$, define $S_\epsilon = \{x \in [0,1] : f(x) \geq M - \epsilon\}$. To have $f(x) \geq M - \epsilon$ the necessary and sufficient condition is to have $f_n(x) \geq M - \epsilon$ for all n. Hence we have $S_\epsilon = \bigcap_{n\geq1} f_n^{-1}([m-\epsilon,\infty))$. Note that S_ϵ is nonempty and closet set for any ϵ. For finite number of positive numbers $\epsilon_i, \bigcap_i S_{\epsilon_i} = S_{min\epsilon_i} \neq \emptyset$. Since $[0,1]$ is compact $\bigcap_\epsilon S_\epsilon \neq \emptyset$. Suppose $t \in \bigcap_\epsilon S_\epsilon$. Then we have, $M - \epsilon \leq f(t) \leq M$.Since $\epsilon > 0$ is arbitrary, $f(t) = M$.
b.) The conclusion does not remain valid if the condition $(*)$ is replaced by *Suppose there exists* n_x *such that* $f_n(x) \geq f_{n+1}(x)$ *for all* $x \in [0,1]$ *and* $n > n_x$. For a counter example consider $f_n(x) = min\{nx, 1-x\}$.

PROBLEM 6. Let (f_n) be a nondecreasing sequence of functions from $[0,1]$ to $[0,1]$. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ and suppose also that f is continuous. Prove that $f_n \to f$ uniformly.

SOLUTION. Since $[0,1]$ is compact and f is continuous on $[0,1], f$ is uniformly continuous. Fix $\epsilon > 0$. Then there is a $\delta > 0$ such that,whenever $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. Choose a natural number N such that $\frac{1}{N} < \delta$. Then for $0 \leq k \leq N$ let $\xi_k = \frac{k}{N}$ and divide $[0,1]$ into subintervals of the form $[\xi_{k-1}, \xi_k]$. Since, $f_n(x) \to f$ pointwise, we can find $M > 0$, by taking maximum over the finite (ξ_k), so that $|f_n(\xi_k) - f(\xi_k)| < \epsilon$ whenever $n \geq M$. Since (f_n) is a nondecreasing sequence of functions for $x \in [\xi_{k-1}, \xi_k]$ we have,

$$f(\xi_{k-1}) - \epsilon < f_n(x) < f(\xi_{k-1}) + 2\epsilon$$

which is equivalent to $|f_n(x) - f(\xi_{k-1})| < 2\epsilon$. Thus, we have the following estimate,

$$|f_n(x) - f(x)| \leq |f_n(x) - f(\xi_{k-1}| + |f(\xi_{k-1}) - f(x)| < 3\epsilon.$$

Since the above inequality does not depend on the particular point, we have the uniform convergence of f_n to f.

PROBLEM 7. Suppose $f : R^n \to R^m$ satisfies following properties.
(i) For all compact subsets K of R^n, $f(K)$ is also compact.
(ii) For decreasing sequence (K_n) of compact subsets of R^n one has $f(\cap_n K_n) = \cap_n f(K_n)$.
Show that f is a continuous function on R^n.

SOLUTION. Pick $\epsilon > 0$ and $x \in R^n$. Let B be an open ball of radius ϵ and centered at $f(x)$ and for all n let K_n be the closed ball of radius $\frac{1}{n}$ centered at x. By (ii) we have $\cap_{n=1}^{\infty} f(K_n) = \{f(x)\}$. For all $n = 1, 2, ...,$ the sets $(R^n - B) \cap f(K_n)$ are compact and these sets form a decreasing sequence. By the previous equality their intersection is the empty set. Hence, there exists an n_0 so that $(R^n - B) \cap f(K_{n_0}) = \emptyset$. From this we get, when $|y - x| < \frac{1}{n_0}$ then $|f(y) - f(x)| < \epsilon$. And this shows that f is continuous at x. Since x was arbitrary, f is continuous on R^n.

PROBLEM 8. Suppose X, Y are metric spaces and $f : X \to Y$ is a continuous function. For all n let K_n be nonempty compact sets such that $K_{n+1} \subset K_n$ and let $K = \cap K_n$. Show that $f(K) = \cap f(K_n)$.

SOLUTION. Since $f(K) \subset f(K_n)$ for all n, we have $f(K) \subset \cap f(K_n)$. Now, let $y \in \cap f(K_n)$. Then, $f^{-1}(\{y\}) \cap K_n$ is non-empty and compact. Also, since for any n,

$$f^{-1}(\{y\}) \cap K_{n+1} \subset f^{-1}(\{y\}) \cap K_n$$

the set

$$\bigcap_{n=1}^{\infty} (f^{-1}(\{y\}) \cap K_n) = f^{-1}(\{y\}) \cap K$$

is non-empty, so $y \in f(K)$.

PROBLEM 9. Show that every orthonormal sequence in an infinite-dimensional Hilbert space converges weakly to 0.

SOLUTION. Let (x_j) be an orthonormal sequence in a Hilbert space H. We need to show that $f(x_j) \to 0$ for all $f \in H^*$. By Riesz representation theorem that is same as showing $< x_j, y > \to 0$ for all $y \in H$. This follows from the Bessel's inequality

$$\sum_j | < x_j, y > |^2 \leq ||y||^2 < \infty.$$

PROBLEM 10. Let X be a topological space, Y a Hausdorff space, and f, g continuous functions from X to Y.

a.) Show that $\{x : f(x) = g(x)\}$ is closed.
b.) Show that if $f = g$ on a dense subset of X, then $f = g$ on all of X.

SOLUTION.
a.) We show that $\{x : f(x) \neq g(x)\}$ is open. Pick any x with $f(x) \neq g(x)$. Since Y is Hausdorff, we can find disjoint open sets U, V with $f(x) \in U$ and $g(x) \in V$. By continuity, $f^{-1}(U)$ and $g^{-1}(V)$ are open sets in X and both contain x, so $O = f^{-1}(U) \cap g^{-1}(V)$ is an open set around x. For any $y \in O$, $f(y) \in U$ and $g(y) \in V$ so $f(y) \neq g(y)$.
b.) The set where $f = g$ is closed. If $f = g$ on any set A, then $f = g$ on closure of A, since the closure of A is the smallest closed set which contains A. "Dense" means that the closure is all of X.

PROBLEM 11. Suppose (f_n) is a sequence of continuous functions such that $f_n : [0,1] \to R$ and as $n, m \to \infty$,

$$\int_0^1 (f_n(x) - f_m(x))^2 dx \to 0.$$

Suppose also that $K : [0,1] \times [0,1] \to R$ is continuous. Define $g_n : [0,1] \to R$ by

$$g_n = \int_0^1 K(x,y) f_n(y) dy.$$

Show that (g_n) is uniformly convergent sequence.

SOLUTION. First, let us show that the sequence (g_n) is cauchy in the supremum norm.

$$|g_n(x) - g_m(x)| \leq \int_0^1 |K(x,y)||f_n(y) - f_m(y)|dy$$
$$\leq \Big(\int_0^1 |K(x,y)|^2 dy\Big)^{\frac{1}{2}} \Big(\int_0^1 |f_n(y) - f_m(y)|^2 dy\Big)^{\frac{1}{2}}$$

implies that

$$\sup_{x\in[0,1]} |g_n(x) - g_m(x)| \leq \sup_{x\in[0,1]} \Big(\int_0^1 |K(x,y)|^2 dy\Big)^{\frac{1}{2}} \Big(\int_0^1 |f_n(y) - f_m(y)|^2 dy\Big)^{\frac{1}{2}}.$$

Since K is continuous it is integrable and $M = \sup_{x,y\in[0,1]} |K(x,y)|$ exists and this gives,

$$||g_n(x) - g_m(x)||_\infty \leq M\Big(\int_0^1 |f_n(x) - f_m(x)|^2 dy\Big)^{\frac{1}{2}} \to 0$$

so, (g_n) is a Cauchy sequence in the supremum norm.But, we know that $C[0,1]$ is complete under this norm, so the sequence (g_n) converges under this norm which is equivalent to converging uniformly.

PROBLEM 12. Let C denote the space of all bounded continuous functions on the real line R equipped with the supremum norm. Let S be the subspace of C consisting of functions f such that

$$\lim_{n\to\infty} f(x)$$

exists.
a.) Is S a closed linear subspace of C?
b.)Show that there is a bounded linear functional L on C so that

$$L(f) = \lim_{x\to\infty} f(x)$$

for all $f \in S$.
c.) Is there a bounded Borel measure μ so that $L(f) = \int_R f d\mu$ for all $f \in C$?

SOLUTION.
a.) Given $\epsilon > 0, \sup_{t\in R} |f_n(t) - f(t)| = ||f_n - f||_\infty < \epsilon/3$ we need to decide whether or not $f \in S$.
Claim: $f \in S$ i.e.$\lim_{n\to\infty} f(x)$ exists. Suppose not, then for all $\delta > 0$, there is an $\epsilon_0 > 0$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$. Now, we are given $||f_n - f||_\infty < \epsilon/3 \Leftrightarrow \sup_{t\in R} |f_n(t)-f(t)| < \epsilon/3$. But then we have $|f_n(x)-f(x)| < \epsilon/3$ and $|f_n(y)-f(y)| < \epsilon/3$. This yields,

$$|f(x) - f(y)| < |f(x) - f_n(x)| + |f_n(y) - f(y)| + |f_n(x) - f_n(y)| < \epsilon_0/3 + \epsilon_0/3 + \epsilon_0/3$$

$\Rightarrow |f(x)-f(y)| < \epsilon_0$, which gives us a contradiction. Thus, S is a closed linear subspace.
b.) We will use Hahn-Banach Theorem to show the existence of L on C. For this, we need to find a sublinear functional p on C such that $\rho_0(f) \leq p(f)$ for all $f \in S$. Define,

p(f)= $\begin{cases} \lim_{x\to\infty} f(x) & \text{if } f \in S \\ 0 & \text{otherwise} \end{cases}$

then, $p(f+g) = \lim_{x\to\infty}(f+g) = p(f)+p(g)$ if $f, g \in S$. and $p(f+g) = 0 = p(f)+p(g)$ if $f, g \notin S$. Also, $p(af) = a\lim_{x\to\infty} f(x) = ap(f)$ if $f \in S$ and $p(af) = 0 = ap(f)$ if $f \notin S$.
So, p is indeed a sublinear functional on C such that $\rho_0(f) = \lim_{x\to\infty} f(x) \leq p(f)$ for all $f \in S$. Therefore, by the Hahn Banach Theorem, there is a linear functional L on C such that $L(f) \leq p(y)$ for all $f \in C$ and $L(f) = \rho_0(f) = \lim_{x\to\infty} f(x)$ if $f \in S$.
c.) We claim that there is no such Borel measure. Let us suppose there is one. Then on the left hand side we will have $\lim_{x\to\infty} f(x)$ which is translation invariant $i.e. \lim_{x\to\infty} f(x) = \lim_{x\to\infty} f(x + a)$ for all $a \in R$. But, the only measure which is translation invariant is the Lebesgue measure, which is not bounded. So, there is no such bounded Borel measure.

PROBLEM 13. Let X be a normed linear space. Show that if S is an open subspace of X, then $S = X$.

SOLUTION. Let S be an open subspace of X. Since S is open and $0 \in S$, there exists $r > 0$ such that $B(0,r) \subset S$. Let $x \in X$. Fix any $R > \frac{1}{r}||x||$, and set $y = \frac{1}{R}x$. Then, $||y|| = \frac{1}{R}||x|| < r$, so $y \in B(0,r) \subset S$. Since S is closed under scalar multiplication, we conclude that $x = Ry \in S$. Thus, we have shown $X \subseteq S$ and the other inclusion is clear so that we have $X = S$.

PROBLEM 14. Suppose that A and K are closed subsets of an additive topological group G, prove that if K is compact then $A + K$ is closed.

SOLUTION. Let $p \in \overline{A+K}$. For each neighborhood U of p, let $K_U = \{k : k \in K, k \in U - A\}$. Since $p \in \overline{A+K}$, each K_U is non-empty. It is clear that if $U_1 \subseteq U_2$, then $K_{U_1} \subseteq K_{U_2}$. It follows that the closed sets $\overline{K_U}$ have the finite intersection property. So their intersection is non-empty. Now let $k_0 \in K = \cap\overline{K_U}$. Thus, if N is any neighborhood of the identity,

$$(N + k_0) \cap (N + p - A) \neq \emptyset.$$

This means that $(N - N + k_0) \cap (p - A) \neq \emptyset$. If M is any neighborhood of the identity, there is a neighborhood N of the identity such that $N - N \subseteq M$. Thus, any neighborhood of k_0 intersects $p - A$. Since A is closed, $p - A$, and thus $p \in A + k_0 \subseteq a + K$.

PROBLEM 15. Let X and Y be normed vector spaces, and let $L : X \to Y$ be linear and bounded.
a.) Show that $N(L) = \{x \in X : L(x) = 0\}$ is a closed subspace of X.
b.) Now let $X = C$, the set of complex numbers. Shiw that $R(L) = \{L(x) : x \in C\}$ is a closed subspace of Y. Hint: Every vector in C is a scalr multiple of 1.

SOLUTION.
a.) Let $x, y \in N(L)$ and $a, b \in F$, then $L(ax+by) = aL(x)+bL(y) = 0$, so $ax+by \in N(L)$. Thus, $N(L)$ is a subspace.
Suppose that x is a limit point of $N(L)$. Then there exists $x_n \in N(L)$ such that $x_n \to x$. But L is continuous, so $0 = L(x_n) = L(x)$, so $x \in N(L)$.
b.) If $p = L(x), q = L(y) \in R(L)$ and $a, b \in F$, then $ap + bq = aL(x) + bL(y) = L(ax + by) \in R(L)$. Thus, $R(L)$ is a subspace.
Suppose that y is a limit point of Y. Then, there exists $y_n = L(x_n) \in R(L)$ such that $y_n \to y$. We have $x_n \in C$ and L is linear, so $L(x_n) = x_n L(1)$. If $L(1) = 0$ then $R(L) = \{0\}$ and we are done. If $L(1) \neq 0$, then

$$||y_n - y_m|| = ||L(x_n) - L(x_m)|| = ||(x_n - x_m)L(1)|| = |x_n - x_m|.||L(1)||.$$

Since $||L(1)||$ is a fixed constant and $\{y_n\}$ is a Cauchy sequence in Y, we conclude that $\{x_n\}$ is a Cauchy sequence of scalars in C, and since L is continuous, this implies $y_n = L(x_n) \to L(x)$. Since limits are unique, we have $y = L(x) \in R(L)$.

PROBLEM 16. Define $L : l^2 \to l^2$ by $L(x_1, x_2, ...) = (x_2, x_3, ...)$. Prove that L is bounded and find $||L||$. Is L injective?

SOLUTION. Let $x = (x_1, x_2, ...) \in l^2$, then

$$||L(x)||_2^2 = \sum_{k=2}^{\infty} |x_k|^2 \leq \sum_{k=1}^{\infty} |x_k|^2 = ||x||_2^2.$$

Hence, L is bounded, and

$$||L|| = \sup_{||x||_2=1} ||L(x)||_2 \leq \sup_{||x||_2=1} ||x||_2 = 1.$$

On the other hand, since $e = (0, 1, 0, ...)$ and $L(e) = (1, 0, 0, ...)$ are both unit vectors, we have $||L|| \geq 1$. Therefore, $||L|| = 1$. L is not injective since $L(1, 0, 0, ...) = L(0, 0, 0, ...)$.

PROBLEM 17. Let X be a normed space, and suppose that $x_n \to x \in X$. Show that there exists a subsequence (x_{nk}) such that

$$\sum_{k=1}^{\infty} ||x - x_{nk}|| < \infty.$$

SOLUTION. We are given that $||x-x_n|| \to 0$. There exists, N_1 such that $||x-x_n|| < \frac{1}{2}$ for $n \geq N_1$. Let $n_1 = N_1$. There exists an N_2 such that $||x-x_n|| < \frac{1}{2^2}$ for $n \geq N_2$. Choose any $n_2 > n_1, N_2$. There exists an N_3 such that $||x - x_n|| < \frac{1}{2^3}$ for $n \geq N_3$. Choose any $n_3 > n_2, N_3$. Continuing in this fashion, we obtain $n_1 < n_2 < n_3 < ...$ in such a way that

$$\sum_{k=1}^{\infty} ||x - x_k|| \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$