SOME PROBLEMS IN REAL ANALYSIS.

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PROBLEM 1. (10 points) Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, ...$; $f_1 \geq f_2 \geq f_3 \geq ... \geq 0$; $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.
a)Give a counterexample to show that we do not have generally the following result. $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.
b) Without changing any other assumptions just add one more assumption and prove that the conclusion is satisfied in this case.

PROBLEM 2. (10 points) Suppose $\mu(X) < \infty$, f_n is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. Show by a counterexample that the conclusion is not valid if we omit $\mu(X) < \infty$.

PROBLEM 3. (10 points)Suppose $f_1 \in L_1(X, \mu)$.Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$.

PROBLEM 4. (10 points)Let X be an uncountable set,let M be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second case.Prove that M is a σ-algebra in X and that μ is a measure on M.

PROBLEM 5. (10 points)Let E_k be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.(*)$$

a.)Then show that almost all $x \in X$ lie in at most finitely many sets E_k.
b.)Is the conclusion still valid if we omit the condition $(*)$?

PROBLEM 6. (10 points)Find a sequence (f_n) of Borel measurable functions on R which decreses uniformly to 0 on R , but $\int f_n dm = \infty$ for all n.Also,find a sequence (g_n) of Borel measurable functions on $[0, 1]$ such that $g_n \to 0$ pointwise but $\int g_n dm = 1$ for all n. (here m is the Lebesgue measure!)

PROBLEM 7. (10 points)Show that Monotone Convergence Theorem can be proved as a corollary of the Fatou's lemma.

PROBLEM 8. (10 points)Let $f \in L^+$ and $\int f < \infty$, then show that the set $\{x : f(x) > 0\}$ is $\sigma-$finite.

PROBLEM 9. (10 points)
a.)If f is nonnegative and integrable on A, then show that

$$\mu(\{x : x \in A, f(x) \geq c\}) \leq \frac{1}{c} \int_A f(x) d\mu$$

b.)If $\int_A |f(x)| d\mu = 0$, prove that $f(x) = 0$ a.e.

PROBLEM 10. (10 points)
a.)Consider a measure space (X, μ) with a finite,positive,finitely additive measure μ. Prove that μ is countably additive if and only if it satisfies the following condition. If A_n is a decresing sequence of sets with empty intersection then

$$\lim_{n \to \infty} \mu(A_n) = 0.$$

b.)Now suppose that X is locally compact Hausdorff space, that Br is the Borel $\sigma-$algebra, and that μ is finite, positive, finitely additive measure on Br.Suppose moreover that μ is regular, that is for each $B \in Br$ we have,

$$\mu(B) = \sup_K \{\mu(K) : K \subseteq B, K - compact\}$$

Prove that μ is countably additive.

PROBLEM 11. (10 points)Let λ be Lebesgue measure on R. Show that for any Lebesgue measurable set $E \subset R$ with $\lambda(E) = 1$, there is a Lebesgue measurable set $A \subset E$ with $\lambda(A) = \frac{1}{2}$.

PROBLEM 12. (10 points)Let m be a countably additive measure defined for all sets in a $\sigma-$algebra M.
a.)If A and B are two sets in M with $A \subset B$, then show that $m(A) \leq m(B)$.
b.)Let (B_n) be any sequence of sets in M.Then show that $m(\cup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} m(B_n)$.

PROBLEM 13. (10 points)
a.)Let (E_n) be an infinite decreasing sequence of Lebesgue measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each n.Let $m(E_1)$ be finite, where m is the Lebesgue meausre.Then show that $m(\cap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} m(E_n)$.
b.)Show by acounterexample that we can not omit the condition $m(E_1)$ is finite.

PROBLEM 14. (10 points)
a.)Show that we may have strict inequality in Fatou's Lemma.
b.)Show that Monotone Convergence Theorem need not hold for decreasing sequences of functions.

PROBLEM 15. (10 points)Let (f_n) be a sequence of nonnegative measurable functions that converge to f, and suppose that $f_n \leq f, \forall n$. Then show that $\int f = \lim \int f_n$.

PROBLEM 16. (10 points)Suppose $A \subset R$ is Lebesgue measurable and assume that

$$m(A \cap (a,b)) \leq \frac{(b-a)}{2}$$

for any $a, b \in R, a < b$.Prove that $m(A) = 0$.

PROBLEM 17. (10 points)Choose $0 < \lambda < 1$ and construct the Cantor set K_λ as follows:Remove from $[0,1]$ its middle part of length λ; we are left with two intervals L_1 and L_2. Remove from each of them their middle parts of length $\lambda|I_i|, i = 1,2$, etc and keep doing this ad infimum. We are left with the set K_λ. Prove that the set K_λ has Lebesgue measure 0.

PROBLEM 18. (10 points)Let $A \subset [0,1]$ measurable set of positive measure.Show that there exist two points $x' \neq x''$ in A with $x' - x''$ rational.

PROBLEM 19. (10 points)Let $f : R^n \to R$ be an arbitrary function having the property that for each $\epsilon > 0$, there is an open set U with $\lambda(U) < \epsilon$ such that f is continuous on $R^n - U$(in the relative topology).Prove that f is measurable.

PROBLEM 20. (10 points)Prove or disprove that composition of two Lebesgue integrable functions with compact support $f, g : R \to R$ is still integable.

PROBLEM 21. (10 points) Let (X, M, μ) be a positive measure space with $\mu(X) < \infty$. Show that a measurable function $f : X \to [0, \infty)$ is integrable (i.e. one has $\int_X f d\mu < \infty$) if and only if the series

$$\sum_{n=0}^{\infty} \mu(\{x : f(x) \geq n\})$$

converges.

PROBLEM 22. (10 points)
a.) Is there a Borel measure μ (positive or complex) on R with the property that

$$\int_R f d\mu = f(0)$$

for all continuous $f : R \to C$ of compact support? Justify.
b.) Is there a Borel measure μ (positive or complex) on R with the property that

$$\int_R f d\mu = f'(0)$$

for all continuous $f : R \to C$ of compact support? Justify.

PROBLEM 23. (10 points) Let f_n be a sequence of real-valued functions in $L^1(R)$ and suppose that for some $f \in L^1(R)$

$$\int_{-\infty}^{\infty} |f_n(t) - f(t)| dt \leq \frac{1}{n^2}, n \geq 1.$$

Prove that $f_n \to f$ almost everywhere with respect to Lebesgue measure.

PROBLEM 24. (10 points) Let m denote the Lebesgue measure on $[0,1]$ and let (f_n) be a sequence in $L^1(m)$ and h a non-negative element of $L^1(m)$. Suppose that
i.) $\int f_n g dm \to 0$ for each $g \in C([0,1])$ and
ii.) $|f_n| \leq h$ for all n.
Show that $\int_A f_n dm \to 0$ for each Borel subset $A \subset [0,1]$.

PROBLEM 25. (10 points)
a.)Prove the Lebesgue Dominated Convergence Theorem.
b.)Here is a version of Lebesgue Dominated Convergence Theorem which is some kind of extension of it.Prove this.
Let (g_n) be asequence of integrable functions which converges a.e. to an integarble function g.Let (f_n) be asequence of measurable functions such that $|f_n| \leq g_n$ and (f_n) converges to f a.e. If $\int g = \lim \int g_n$, then $\int f = \lim \int f_n$.
c.) Show that under hypotheses of the part b.) we have $\int |f_n - f| \to 0$ as $n \to \infty$.
d.)Let (f_n) be asequence of integrable functions such that $f_n \to f$ a.e. with f is integrable.Then show that
$\int |f - f_n| \to 0$ as $n \to \infty$ if and only if $\int |f_n| \to \int |f|$ as $n \to \infty$.

PROBLEM 26. (10 points) Evaluate

$$\lim_{n \to \infty} \int_0^n (1 + \frac{x}{n})^n e^{-2x} dx,$$

justifying any interchange of limits you use.

PROBLEM 27. (10 points)
a.) Let (a_n) be a sequence of nonnegative real numbers. Set $\mu(\emptyset) = 0$, and for every nonempty subset A of N (set of natural numbers) set $\mu(A) = \sum_{n \in A} a_n$.Show that the set function $\mu : P(N) \to [0, \infty]$ is a measure.
b.) Let X be a nonempty set and let $f : X \to [0, \infty]$ be a function.Define μ by $\mu(A) = \sum_{a \in A} f(x)$ if $A \neq \emptyset$ and is at most countable, $\mu(A) = \infty$ if A is uncountable, and $\mu(\emptyset) = 0$.Show that μ is a measure.

PROBLEM 28. (10 points) Let F be a nonempty collection of subsets of a set X and let $f : F \to [0, \infty]$ be a function. Define $\mu : P(X) \to [0, \infty]$ by $\mu(\emptyset) = 0$ and

$$\mu(A) = \inf\{\sum_{n=1}^{\infty} f(A_n) : (A_n) \subseteq F, and, A \subseteq \cup_{n=1}^{\infty} A_n\}$$

for each $A \neq \emptyset$, with $\inf \emptyset = \infty$. Show that μ is an outer measure.

PROBLEM 29. (10 points) Let $f : R \to R$ be a Lebesgue integrable function.Show that

$$\lim_{t \to \infty} \int f(x) cos(xt) d\lambda(x) = \lim_{t \to \infty} \int f(x) sin(xt) d\lambda(x) = 0$$

PROBLEM 30. (10 points) For a sequence (A_n) of subsets of a set X define $\liminf A_n = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$ and $\limsup A_n = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i$
Now let (X, S, μ) be a measure space and let (E_n) be a sequence of measurable sets.Show the following:
a.) $\mu(\liminf E_n) \leq \liminf \mu(E_n)$
b.) If $\mu(\cup_{n=1}^{\infty} E_n) < \infty$, then $\mu(\limsup E_n) \geq \liminf \mu(E_n)$

PROBLEM 31. (10 points)
a.) Let X be a nonempty set and let δ be the Dirac measure on X with respect to a point. Show that every function $f : X \to R$ is integrable and that $\int f d\delta = f(a)\delta(a) = f(a)$.
b.) Let μ be the counting measure on N(set of natural numbers). Show that a function $f : N \to R$ is integrable if and only if $\sum_{n=1}^{\infty} |f(n)| < \infty$. Also, show that in this case $\int f d\mu = \sum_{n=1}^{\infty} f(n)$.

PROBLEM 32. (10 points) Let (X, S, μ) be a measure space and let $f_1, f_2, f_3, ...$ be nonnegative integrable functions such that $f_n \to f$ a.e. and $\lim \int f_n d\mu = \int f d\mu$. If E is a measurable set, then show that $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$.

PROBLEM 33. (10 points) Let $f : [0, \infty) \to R$ be a continuous function such that $f(x + 1) = f(x)$ holds for all $x \geq 0$. If $g : [0, 1] \to R$ is an arbitrary continuous function, then show that

$$\lim_{n\to\infty} \int_0^1 g(x)f(nx)dx = (\int_0^1 g(x)dx)(\int_0^1 f(x)dx).$$

PROBLEM 34. (10 points) Show that

$$\int_0^{\infty} \frac{sin^2(x)}{x^2} dx = \frac{\pi}{2}.$$

PROBLEM 35. (10 points)
a.)Let (f_n) be a sequence of measurable functions and let $f : X \to R$. Assume that

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0(*)$$

holds for every $\epsilon > 0$. Show that f is measurable.
b.) Assume that $(f_n) \subseteq M$ satisfies $f_n \uparrow$ and $f_n \to^{\mu} f$(i.e. f_n goes to f in measure). Show that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

c.) Assume that $(f_n) \subseteq M$ satisfies $f_n \geq 0$ a.e. and $f_n \to^{\mu} f$(i.e. f_n goes to f in measure). Show that $f \geq 0$ a.e.

PROBLEM 36. (10 points) Let g be an integrable function and let (f_n) be a sequence of integrable functions such that $|f_n| \leq g$ a.e. holds for all n. Suppose that $f_n \to^{\mu} f$ (i.e.f_n goes to f in measure), then show that f is an integrable function and $\lim \int |f_n - f| d\mu = 0$.

PROBLEM 37. (10 points) Let f be a.e. positive measurable function and let

$$m_i = \mu(\{x \in X : 2^{i-1} < f(x) \leq 2^i\})$$

for each integer i. Show that f is integrable if and only if $\sum_{-\infty}^{\infty} 2^i m_i < \infty$.

PROBLEM 38. (10 points)
a.) Let $f \in L_1(\mu)$(i.e. f is integrable) and let $\epsilon > 0$. Show that

$$\mu(\{x \in X : |f(x)| \geq \epsilon\}) \leq \epsilon^{-1} \int |f| d\mu$$

b.) If $f_n \to f$ in $L_1(\mu)$ then show that $f_n \to f$ in measure.

PROBLEM 39. (10 points) Suppose f is integrable on a set A. Then, show that given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|\int_E f(x) d\mu| < \epsilon$$

for every measurable set $E \subset A$ of measure less than δ.

PROBLEM 40. (10 points) Suppose f is integrable function on $R(\Leftrightarrow f \in L^1(R))$.Then, show that

$$\lim_{t \to 0} \int |f(t+x) - f(x)| dx = 0.$$

PROBLEM 41. (10 points) Show that every extended real valued measurable function f is the limit of a sequence (f_n) of simple functions.

PROBLEM 42. (10 points) Suppose μ is a probability measure on X $i.e.\mu(X) = 1$.Let $A_1, A_2, A_3, ... \in U$ be sets in the $\sigma-$algebra U such that $\sum_{i=1}^n \mu(A_i) > n - 1$. Show that $\mu(\cap_{k=1}^n A_k) > 0$.

PROBLEM 43. (10 points) Suppose f is an integrable function on $X = R^p$.
i) Show that $\forall \epsilon > 0$, there exists a measurable set with finite measure such that f is bounded on A and $\int_{(X-A)} |f| d\mu < \epsilon$
ii) From this deduce that

$$\lim_{\mu(E) \to 0} \int_E |f| d\mu = 0.$$

PROBLEM 44. (10 points)
i) Show that an algebra A is a $\sigma-$algebra if and only if it is closed under countable increasing unions(i.e. If $(E_j)_{j=1}^{\infty} \subset A$ and $E_1 \subset E_2 \subset E_3 \subset ...$, then $\bigcup_{j=1}^{\infty} E_j \in A$.)
ii) Suppose $\mu_1, \mu_2, \mu_3, ...$ are measures on (X, M) and $a_1, a_2, a_3, ... \in [0, \infty)$. Show that $\sum_1^n a_j \mu_j$ is a measure on (X, M).

PROBLEM 45. (10 points)
i) Let (X, M, μ) be a measure space and $E, F \in M$. Show that

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

ii) Given a measure space (X, M, μ) and $E \in M$, define $\mu_E(F) = \mu(A \cap E)$. Show that μ_E is a measure.

PROBLEM 46. (10 points) We know that if μ is a measure(in the sense that it is countably additive) then it is continuous from below. {i.e. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq$ ...then $\mu(\bigcup A_i) = \lim \mu(A_i)$}.Similarly, we know that if $\mu(X)$ is finite then μ is continuous from above.{i.e. If ... $\subseteq A_3 \subseteq A_2 \subseteq A_1$ then $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap A_n)$.}. Show that a finitely additive measure is a measure if and only if it is continuous from below.Now suppose that $\mu(X) < \infty$. Show that μ is a measure if and only if it is continuous from above.Give a counterexample to show that if μ is a measure with $\mu(X) = \infty$ then it is not necessarily continuous from above.

PROBLEM 47. (10 points) Let μ^* be an outer measure on X and let $(A_j)_{j=1}^{\infty}$ be a sequence of disjoint μ^*- measurable sets(in the sense of Carethedory). Show that $\mu^*(E \cap (\bigcup_1^{\infty} A_j)) = \sum_1^{\infty} \mu^*(E \cap A_j)$ for any $E \subset X$.

PROBLEM 48. (10 points) Let $f : R^n \to R$ be a Lebesgue measurable function such that

$$m(\{x : |f(x)| > t\}) \leq \frac{c}{t^2}, t > 0.$$

Prove that there exists a constant C_1 such that for any Borel set $E \subset R^n$ of finite and positive measure

$$\int_E |f(x)| dx \leq C_1 \sqrt{m(E)}.$$

PROBLEM 49. (10 points) Let $B(m, 1)$ be $m-$dimensional ball of radius 1 centered at the origin in R^m.
a.) Show that there exists a function $f : R \to [0, 1]$ such that

$$m(B(n + 1, 1)) = m(B(n, 1)) \int [f(t)]^n dt.$$

Here m denotes the Lebesgue measure.
b.) Show that $\int [f(t)]^n dt \to 0$ as $n \to \infty$.
c.) Show that for any positive number A, $A^n m(B(n, 1)) \to 0$ as $n \to \infty$.

PROBLEM 50. (10 points) Let f be an integrable function. Show that
i) $\mu(\{x : |f(x)| \geq a\}) \leq \frac{1}{a} \int |f| d\mu$.
ii) $\mu(\{x : |f(x)| \geq a\}) = o(\frac{1}{a})$ as $a \to \infty$.

PROBLEM 51. (10 points) Let $\big(X, M\big)$ be a measure space. If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$, for $E \in M$. Show that λ is a measure on M, and for any $g \in L^+$, $\int g d\lambda = \int f g d\mu$.

PROBLEM 52. (10 points) Let $f(x) = x^{-1/2}$ if $0 < x < 1, f(x) = 0$ otherwise. Let $(r_n)_1^\infty$ be an enumeration of rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.Show that
a.) $g \in L^1(m)$ and in particular $g < \infty$ a.e.$\big($ Here m is the Lebesgue measure $\big)$.
b.) $g^2 < \infty$ a.e. but g^2 is not integrable on any interval.
c.) g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

PROBLEM 53. (10 points) Compute the following limits and justify the calculations:
a.) $\lim_{n\to\infty} \int_0^\infty \big(1 + \frac{x}{n}\big)^{-n} sin(\frac{x}{n}) dx$;
b.) $\lim_{n\to\infty} \int_0^1 \big(1 + nx^2\big)\big(1 + x^2\big)^{-n} dx$;
c.) $\lim_{n\to\infty} \int_0^\infty n sin(\frac{x}{n}) \big[x(1 + x^2)\big]^{-1} dx$;
d.) $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1} dx$, $(a \in R)$

PROBLEM 54. (10 points) Suppose f_n and f are measurable complex-valued functions and $\phi : C \to C$.
a.) If ϕ is continuous and $f_n \to f$ a.e., then show that $\phi \circ f_n \to \phi \circ f$ a.e.
b.) Show that if ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$, uniformly, almost uniformly, or in measure, respectively.
c.) Give counterexamples when the continuity assumptions on ϕ are not satisfied.

PROBLEM 55. (10 points) Suppose $f_n \to f$ almost uniformly, then show that $f_n \to f$ a.e. and in measure.

PROBLEM 56. (10 points) Show that if $f : [a, b] \to C$ is Lebesgue measurable and $\epsilon > 0$, then there is a set $E \subset [a, b]$ such that $m(E^c) < \epsilon$ and $f|_E$ is continuous. Moreover, E may be taken to be compact.

PROBLEM 57. (10 points) A measure μ is called *semi-finite* if every set of infinite measure contains a subset of finite, nonzero measure. Show that every $\sigma-$finite measure is semi-finite.

PROBLEM 58. Let μ be the counting measure on N. Prove that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

PROBLEM 59. Prove that for $a > 0$,

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(ax) dx = \sqrt{\pi} e^{-\frac{a^2}{4}}.$$

PROBLEM 60. Let (q_n) be an enumeration of rationals in $[0, 1]$. Define the function f on $[0, 1]$ by,

$$f(x) = \sum_{n, so, q_n < x} 2^{-n}.$$

(a.) Where is this function continuous/discontinuous?

(b.) Is this function Riemann integrable?

(c.) Is this function Lebesgue integrable?

PROBLEM 61. If f and g are two continuous functions on a common open set in R^n that agree everywhere on the complement of a set of zero Lebesgue measure,then, show that in fact f and g agree everywhere.