2.5 #15. Suppose that $N$ is a normal subgroup of a group $G$, and $f : G \to G'$ is a homomorphism of $G$ onto $G'$. Prove that $f(N)$ is a normal subgroup of $G'$.

Solution

First we must prove that $f(N)$ is a subgroup, and then we must prove that it is normal. Remember that $f(N)$ is the direct image of $N$, which means that

$$f(N) = \{f(n) : n \in N\}.$$

Suppose that $x$, $y$ belong to $f(N)$. Then, by definition of $f(N)$, we must have $x = f(m)$ and $y = f(n)$ for some $m$, $n \in N$. Since $N$ is a subgroup, we know that $mn$ belongs to $N$. Hence $f(mn) \in f(N)$, again by the definition of $f(N)$. Since f is a homomorphism, we therefore have

$$xy = f(m)f(n) = f(mn) \in f(N).$$

Hence $f(N)$ is closed under compositions.

Now suppose that $x$ belongs to $f(N)$. This means that $x = f(n)$ for some $n \in N$. Since $N$ is a subgroup, we have $n^{-1} \in N$, and therefore $f(n^{-1}) \in f(N)$. Since $f$ is a homomorphism, we therefore have

$$x^{-1} = f(n)^{-1} = f(n^{-1}) \in f(N).$$

Thus $f(N)$ is closed under inverses. Since $f(N)$ is nonempty (WHY?), we conclude that $f(N)$ is indeed a subgroup of $G'$.

Now we show that $f(N)$ is normal. Suppose that $a$ is any element of $G'$. We must show that $a\,f(N)\,a^{-1} \subseteq f(N)$. By definition,

$$a\,f(N)\,a^{-1} = \{axa^{-1} : x \in f(N)\}.$$

Therefore, our task is to show that if $x$ is any element of $f(N)$, then $axa^{-1}$ is an an element of $f(N)$.

So, let $x$ be any element of $f(N)$. Then, by definition, $x = f(n)$ for some $n \in N$. Also, since $f$ is onto, we know that $a = f(g)$ for some $g \in G$. Using the fact that $N$ is normal, we see that

$$gng^{-1} \in gNg^{-1} = N.$$

Since $gng^{-1} \in N$, we therefore have $f(gng^{-1}) \in f(N)$. Finally, using the fact that $f$ is a homomorphism, it follows that

$$axa^{-1} = f(g)f(n)f(g)^{-1} = f(gng^{-1}) \in f(N).$$

Thus, we have shown that every element of $a\,f(N)\,a^{-1}$ belongs to $f(N)$, so we conclude that $a\,f(N)\,a^{-1} \subseteq f(N)$. Therefore $f(N)$ is normal. $\square$