Problem 2.5 #18. Let H be a subgroup of a group G, and define

$$N = \bigcap_{a \in G} aHa^{-1}.$$

Prove that N is a normal subgroup of G.

Solution

First we must prove that N is a subgroup, and then we must prove that it is normal. Note that $e = aea^{-1} \in aHa^{-1}$ for every $a \in G$, so $e \in N$. This shows that N is nonempty.

Suppose that x, y belong to N, and let a be any particular element of G. Then, by the definition of N, we have $x \in aHa^{-1}$ and $y \in aHa^{-1}$, so $x = aha^{-1}$ and $y = aka^{-1}$ for some elements h, $k \in H$. Since H is closed under composition we have $hk \in H$, and therefore

$$xy = (aha^{-1})(aka^{-1}) = a(hk)a^{-1} \in aHa^{-1}.$$

This is true for every element a in G, so it follows that

$$xy \in \bigcap_{a \in G} aHa^{-1} = N.$$

Therefore N is closed under compositions.

Now suppose that x belongs to N, and let a be any particular element of G. By definition, $x \in aHa^{-1}$, so $x = aha^{-1}$ for some $h \in H$. As H is closed under inverses, we have $h^{-1} \in H$, and therefore

$$x^{-1} = (aha^{-1})^{-1} = ah^{-1}a^{-1} \in aHa^{-1}.$$

This is true for every element a in G, so

$$x^{-1} \in \bigcap_{a \in G} aHa^{-1} = N.$$

Therefore N is closed under inverses.

So far, we have shown that N is a subgroup, and now we will show that it is normal. Choose any element $b \in G$. We need to show that $bNb^{-1} \subseteq N$. To do this, we have to show that

$$bNb^{-1} \subseteq aHa^{-1} \quad \text{for every } a \in G. \tag{1}$$

To prove equation (1), let a be any particular element of G. We need to show that $bNb^{-1} \subseteq aHa^{-1}$, so choose any $x \in bNb^{-1}$. Then $x = bnb^{-1}$ for some $n \in N$. Since $b^{-1}a$ is just one of the elements of G, we have

$$n \in N \subseteq (b^{-1}a)H(b^{-1}a) = b^{-1}aHa^{-1}b.$$

Therefore

$$n = b^{-1}aha^{-1}b \quad \text{for some } h \in H.$$

Consequently

$$x = bnb^{-1} = b(b^{-1}aha^{-1}b)b^{-1} = aha^{-1} \in aHa^{-1}.$$

As x is an arbitrary element of bNb^{-1}, this shows that

$$bNb^{-1} \subseteq aHa^{-1}. \tag{2}$$

This is exactly what we said that we needed to prove. Or, to put it another way, we have proved that equation (2) hold for every element $a \in G$, so by taking the intersection over all such elements a (and keeping in mind that b is one *fixed* element of G), we see that

$$bNb^{-1} \subseteq \bigcap_{a \in G} aHa^{-1} = N.$$

Therefore N is normal. $\square$