MATH 4107 (Prof. Heil) PRACTICE PROBLEMS WITHOUT SOLUTIONS Spring 2018

Here are a few practice problems on groups. The first ones are easier and the later ones are harder.

1. Let $A(\mathbb{R})$ be the set of all bijections of the real line $\mathbb{R}$ onto itself. We know that this is a group under the operation of composition of functions (you do not need to prove that fact).

(a) Identify three different functions that belong to $A(\mathbb{R})$, i.e., give a formula for three functions that belong to $A(\mathbb{R})$.
Note: You don't need to prove that your functions are bijections, you just need to give formulas for three functions that are bijections.

(b) Identify the identity element of $A(\mathbb{R})$, i.e., give the formula for the function that is the identity of $A(\mathbb{R})$. Prove that your function is indeed the identity element of $A(\mathbb{R})$.

(c) Let $f(x) = x + 1$. Determine (with proof) whether f has finite order or not.

2. Let G and H be groups, and let $f\colon G \to H$ be a homomorphism.

(a) Suppose that $x \in G$. Let k be the order of x, and let m be the order of $f(x)$. Without appealing to any theorems on order, show directly that m divides k.
Hint: $k = mj + r$.

(b) Show that if $|G|$ and $|H|$ are relatively prime (no common divisors) then $\ker(f) = G$.

3. Suppose that M is a normal subgroup of a group G, and N is a normal subgroup of a group H. Then the set $M \times N$ is a normal subgroup of $G \times H$ (you do not need to prove this).

(a) Define $f\colon G \times H \to (G/M) \times (H/N)$ by

$$f\big((a,b)\big) \;=\; (Ma, Nb), \qquad (a,b) \in G \times H.$$

Show that f is a surjective homomorphism.

(b) Use the First Homomorphism Theorem to show that

$$(G \times H)/(M \times N) \;\cong\; (G/M) \times (H/N).$$

4. Suppose that G is a finite abelian group with order $|G| > 1$. Suppose there exists a prime number p such that for each $a \in G$ there exists a positive integer k such that $a^{p^k} = e$ (the integer k depends on the element a). Show that $|G| = p^n$ for some integer n.

5. Suppose that M, N are normal subgroups of a group G, and $M \cap N = \{e\}$. Prove that MN is a normal subgroup of G, and $M \times N \cong MN$.

6. Suppose that M, N are normal subgroups of a group G, and $MN = G$.

(a) Define $f\colon G \to G/M \times G/N$ by $f(a) = (Ma, Na)$ for $a \in G$. Prove that f is a surjective homomorphism of G onto $G/M \times G/N$.

Hint: Surjective is the hard part, do all the other parts of this problem first.

(b) Prove that $\ker(f) = M \cap N$.

(c) Given the results from parts (a) and (b), what does the First Homomorphism Theorem now imply?

7. Let G be a group. Let C be the set of all commutators of elements of G, i.e.,

$$C = \{xyx^{-1}y^{-1} : x, y \in G\}.$$

Unfortunately, C need not be a subgroup of G. Therefore we define the *commutator subgroup* C' to be the subgroup "generated by" C. Specifically, this means that C' is the intersection of all subgroups of G that contain C:

$$C' = \bigcap\{H : H \text{ is a subgroup of } G \text{ and } C \subseteq H\}.$$

(a) Prove that C' is a normal subgroup of G (prove both that C' is a subgroup, and that it is normal).

(b) Prove that G/C' is abelian.