Here are a few practice problems on groups. **You should first work through these WITHOUT LOOKING at the solutions!** After you write your own solution, you can compare to my solution. Your solution does not need to be identical to mine—there are often many ways to solve a problem—but it does need to be CORRECT.

1. Suppose that $H$ is a subgroup of $S_n$ and $H$ contains every 2-cycle $(i_1\ i_2)$. Show that $H = S_n$. In other words, we say that *the 2-cycles generate* $S_n$.

Hint: Show first that $H$ contains every 3-cycle, then every 4-cycle, etc., up to the $n$-cycles. Then show that $H$ contains every permutation, whether it is a cycle or not.

Solution

Every 3-cycle is a product of 2-cycles, because $(i_1\ i_2\ i_3) = (i_1\ i_3)(i_1\ i_2)$. Since $H$ contains every 2-cycle and is closed under composition, it therefore contains every 3-cycle.

Every 4-cycle is a product of a 3-cycle and a 2-cycle, because $(i_1\ i_2\ i_3\ i_4) = (i_1\ i_4)(i_1\ i_2\ i_3)$. Since $H$ contains every 2-cycle and every 3-cycle and is closed under composition, it therefore contains every 4-cycle.

Continuing in this way (more precisely, by doing a proof by induction), we conclude that $H$ contains every $n$-cycle. Since every permutation can be written as a product of cycles and since $H$ is closed under compositions, we conclude that $H$ contains every permutation. $\square$

2. Show that $N = \{e,\ (123),\ (132)\}$ is a normal subgroup of $S_3$.

Solution

We can check this by direct calculation. Using the facts that $(123)^{-1} = (132)$, $(12)^{-1} = (12)$, $(13)^{-1} = (13)$, and $(23)^{-1} = (23)$, we compute that:

$$\begin{aligned}
eNe^{-1} &= \{eee^{-1},\ e(123)e^{-1},\ e(132)e^{-1}\} = \{e,\ (123),\ (132)\},\\
(123)N(123)^{-1} &= \{(123)e(123)^{-1},\ (123)(123)(123)^{-1},\ (123)(132)(123)^{-1}\} = \{e,\ (123),\ (132)\},\\
(132)N(132)^{-1} &= \{(132)e(132)^{-1},\ (132)(123)(132)^{-1},\ (132)(132)(132)^{-1}\} = \{e,\ (132),\ (123)\},\\
(12)N(12)^{-1} &= \{(12)e(12)^{-1},\ (12)(123)(12)^{-1},\ (12)(132)(12)^{-1}\} = \{e,\ (123),\ (132)\},\\
(13)N(13)^{-1} &= \{(13)e(13)^{-1},\ (13)(123)(13)^{-1},\ (13)(132)(13)^{-1}\} = \{e,\ (132),\ (123)\},\\
(23)N(23)^{-1} &= \{(23)e(23)^{-1},\ (23)(123)(23)^{-1},\ (23)(132)(23)^{-1}\} = \{e,\ (123),\ (132)\}.
\end{aligned}$$

Thus for every permutation $\alpha \in S_3$, we have $\alpha N \alpha^{-1} = N$, so $N$ is normal.

Actually, there's a much easier proof: Since $N$ has 3 elements and $S_3$ has 6, the index of $N$ is 2, and therefore $N$ must be normal. $\square$

3. Let $S_n$ be the set of all permutations of $\{1, 2, \dots, n\}$. Show that if $n$ is any integer larger than 2, then there exist two functions $f$, $g \in S_n$ which do not commute.

Solution

Define $f\colon \{1, \dots, n\} \to \{1, \dots, n\}$ by $f(1) = 2$, $f(2) = 1$, and $f(k) = k$ for $k = 3, 4, \dots, n$. Define $g\colon \{1, \dots, n\} \to \{1, \dots, n\}$ by $g(2) = 3$, $g(3) = 2$, and $f(k) = k$ for $k = 1, 4, 5, \dots, n$. Then $f$ and $g$ are elements of $S_n$.

We compute that

$$(f \circ g)(1) = f(g(1)) = f(1) = 2 \qquad \text{and} \qquad (g \circ f)(1) = g(f(1)) = g(2) = 3.$$

Since these are different, the functions $f \circ g$ and $g \circ f$ have different rules, i.e., $f \circ g \neq g \circ f$. □

4. If $\alpha$ is an $r$-cycle, must $\alpha^k$ be an $r$-cycle?

Solution

Sometimes, but not always. For example, if $\alpha$ is an $r$-cycle and $k = r$ then $\alpha^k = e$, which is a 1-cycle but not an $r$-cycle. And here is a less trivial example: $(1\ 2\ 3\ 4)^2 = (1\ 3)\,(2\ 4)$. Thus the square of this 4-cycle is not even a cycle at all, it is a product of two disjoint transpositions. On the other hand, you CAN show that if $r = p$ is PRIME, then $\alpha^k$ is always a cycle. In this case, if $p$ divides $k$ then $\alpha^k = e$, and if $p$ does not divide $k$ then $\alpha^k$ is also a $p$-cycle. □

5. Give an example of $\alpha$, $\beta$, $\gamma \in S_5$ such that $\alpha\beta = \beta\alpha$ and $\alpha\gamma = \gamma\alpha$, but $\beta\gamma \neq \gamma\beta$.

Solution

Try $\alpha = (1\ 2)$, $\beta = (3\ 4)$, and $\gamma = (3\ 5)$. Then $\alpha$ commutes with $\beta$ because they are disjoint, and similarly $\alpha$ commutes with $\gamma$ because they are disjoint, yet $\beta\gamma = (3\ 5\ 4) \neq (3\ 4\ 5) = \gamma\beta$. □

6. Let $\alpha$ be any permutation in $S_n$ (not necessarily a cycle), and let $i$ be a fixed number between 1 and $n$. Prove that $\alpha$ moves $i$ if and only if $\alpha^{-1}$ moves $i$.

Solution

Suppose that $\alpha(i) = j$ where $j \neq i$. Then we know that $\alpha^{-1}(j) = i$. Therefore we can't also have $\alpha^{-1}(i) = i$, because if we did then $\alpha^{-1}$ would not be a bijection. Hence $\alpha^{-1}$ must move $i$.

The converse direction, showing that if $\alpha^{-1}$ moves $i$ then $\alpha$ moves $i$, is entirely similar. □