MATH 4107 (Prof. Heil) PRACTICE PROBLEMS WITHOUT SOLUTIONS Spring 2018

Here are a few practice problems.

1. Let $n \geq 3$ be a fixed integer.

(a) Suppose that $\sigma \in S_n$ has order 2. Prove that σ is a product of one or more *disjoint* transpositions.

(b) Suppose that σ is a k-cycle. Is σ even or odd? (The answer depends on k.)

(c) Show that if H is a subgroup of S_n and every 2-cycle belongs to H, then $H = S_n$.

(d) Show that if σ, τ are two permutations in S_n then σ and $\tau\sigma\tau^{-1}$ have the same order.

(e) Show that if σ, τ are two permutations in S_n and σ is a cycle, then $\tau\sigma\tau^{-1}$ is a cycle.

2. (a) Show that if F is a field, then the only polynomials in $F[x]$ that have multiplicative inverses are nonzero constant polynomials.

(b) Give an example of a commutative ring R with identity such that there exists a nonconstant polynomial $p \in R[x]$ that has a multiplicative identity in $R[x]$.
Hint: Consider $\mathbb{Z}_n[x]$ where n is not prime, and consider "easy" polynomials p (those with degree 1).

3. Let I be a nontrivial ideal in the ring of integers $\mathbb{Z}$. Show that there exists an $n > 1$ such that $I = n\mathbb{Z}$.

4. Let $F = \mathbb{Z}_{11}$, the integers mod 11. Let $p(x) = x^2 + 1$. Show that p is irreducible in $\mathbb{Z}_{11}[x]$, and $\mathbb{Z}_{11}[x]/(p)$ is a field that contains exactly 121 elements.

5. Suppose that R is an integral domain, and let $R[x]$ be the ring of polynomials over R. Prove that $R[x]$ is an integral domain, but is not a field. Prove that $F[x]$ is not a field even if F is a field.

6. Work out the rules for addition and multiplication in the ring $\mathbb{R}[x]/(f)$, where $f(x) = x^3 - 1$. Note that the quotient ring consists of elements $a + bx + cx^2 + (f)$. Is $\mathbb{R}[x]/(f)$ a field? Is (f) a maximal ideal in $\mathbb{R}[x]/(f)$?

7. Let R be a commutative ring. An element $a \in R$ is *nilpotent* if $a^k = 0$ for some positive integer k (k can depend on a). Let N be the set of all nilpotent elements, i.e.,

$$N \;=\; \{a \in R : a^k = 0 \text{ for some } k > 0\}.$$

Note: The Binomial Theorem holds for commutative rings, i.e., if $x,\, y \in R$ and $n > 0$ then

$$(x+y)^n \;=\; \sum_{m=0}^{n} \binom{n}{m} x^m\, y^{n-m}.$$

(a) Suppose that $a^k = 0$ and $b^j = 0$. Show that $(a+b)^{j+k} = 0$.

(b) Show that N is an ideal in R.
Note: You have to prove both the subgroup property and the ideal property.

(c) Suppose that S is an integral domain and $\varphi\colon R \to S$ is a ring homomorphism. Show that $N \subseteq \ker(\varphi)$.

(d) Show that R/N has no nilpotent elements other than the zero element of R/N (which is N). That is, show that if $a \in R$ and $(a+N)^k = N$ for some $k > 0$ then $a + N = N$.

8. Let $F = \mathbb{Z}_p$ where p is prime, and suppose that $q(x)$ is an irreducible polynomial in $F[x]$ of degree n. Then we know that $F[x]/(q)$ is a field.

(a) Suppose that $f,\, g,\, r \in F[x]$ and $f = gq + r$. Show that $f + (q) = r + (q)$.

(b) Show that $F[x]/(q)$ has order p^n.

9. Let F be field, and suppose that $\varphi\colon F[x] \to F[x]$ is an isomorphism that satisfies $\varphi(a) = a$ for every constant polynomial a. Show that there exist $b,\, c \in F$ with $b \neq 0$ such that $\varphi(p)(x) = p(bx + c)$ for all $p \in F[x]$.