1.4 A(S) The Symmetric Group

Definition

Let S be a set. Then

$$A(S) = \{ f : f \text{ is a bijection of } S \text{ onto } S \}.$$

Example

If S is the unit square, then the rotations & flips we discussed before are some of the bijections of the square onto itself. A bijection doesn't have to be a rigid motion, & it doesn't have to be "continuous".

Exercise: Give examples of other bijections of S onto itself.

Definition

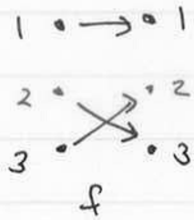
If $S = \{1, 2, \ldots, n\}$, then the symmetric group of degree n is

$$S_n = A(S) = \{ f : f \text{ is a bijection of } \{1, 2, \ldots, n\} \text{ onto itself} \}.$$

Examples of bijections of $\{1,2,3\}$ onto itself.

Two-line notation — Another notation

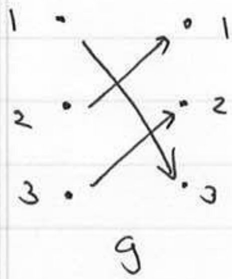
Cycle notation — Permutation notation



$$f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$f = (2\,3)$

or $f = (1)(2\,3)$



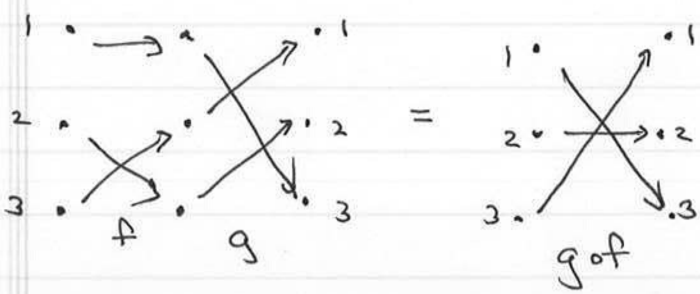
$$g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$g = (1\,3\,2)$

Written more clearly:
$g = (1\ 2\ 3)$
We put a space between each number.

We'll cover cycle notation in more detail later.

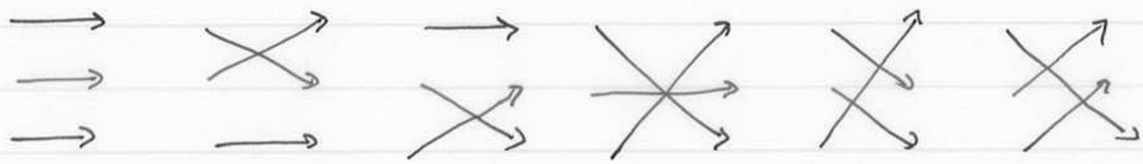
Composition: Compute $f \circ g$



$$g \circ f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{or} \quad g \circ f = (1\,3) = (1\,3)(2)$$

Exercise: $f \circ g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\quad f \circ g = (1\,2) = (1\,2)(3)$

S_3 has 6 elements



i $\quad (1\,2) \quad (2\,3) \quad (1\,3) \quad (1\,2\,3) \quad (1\,3\,2)$

Exercise: S_n has $n!$ elements.

Properties of $A(S)$

a. $A(S)$ is closed under compositions: $f, g \in A(S) \Rightarrow f \circ g \in A(S)$

b. Composition is associative: $(f \circ g) \circ h = f \circ (g \circ h) \quad \forall f, g, h \in A(S)$

c. $\exists$ identity element i s.t. $i \circ f = f \circ i \quad \forall f \in A(S)$

d. $\exists$ inverses: $f \in A(S) \Rightarrow f^{-1} \in A(S)$ &

$$f \circ f^{-1} = i = f^{-1} \circ f.$$

$A(S)$ is an example of a group.

S_n is a finite group.

Shorthand [redacted] notation

We often write fg for $f \circ g$, & $f^2 = f \circ f$, etc.

Note

$fg \neq gf$ in general

$(fg)^2 \neq f^2 g^2$ in general.

Exercise

$fg = fh \implies g = h$

$fg = hg \implies f = h$

Beware: $fg = gh \not\Rightarrow f = h$!