4.4 Maximal Ideals

Definition

A ring R is *simple* if it has no nontrivial ideals, i.e., $\{0\}$ & R are the only ideals.

Definition

An ideal M in a ring R is a *maximal ideal* if

a. $M \subsetneq R$

b. There are no ideals I with $M \subsetneq I \subsetneq R$.

Note: Thus

R is a simple ring $\iff$ $\{0\}$ is a maximal ideal.

Theorem

Let M be a proper ideal in a ring R. Then:

$$M \text{ is maximal } \iff R/M \text{ is simple.}$$

Proof:

$\Leftarrow$ Suppose that R/M is simple. Suppose that I is an ideal in R & $M \subseteq I \subseteq R$.

Consider the canonical map

$$\pi : R \longrightarrow R/M$$
$$\pi(a) = a + M.$$

Consider

$$J = \pi(I) = \{a+M : a \in I\}.$$

Exercise: Show J is an ideal in R/M.

But R/M is simple, so either $J = \{M\}$ (since M is the zero element of R/M), or $J = R/M$.

Case 1: $J = \{M\}$.

We claim that this implies that $I = M$. To see this, suppose that $a \in I$. Then, by definition of J, we have $a + M \in J$. But $J = \{M\}$, so this implies $a + M = M$. Hence $a \in M$, so we have shown that $I \subseteq M$.

Exercise: Show that $M \subseteq I$.

Thus, in this case we have $I = M$.

Case 2: $J = R/M$.

We claim that this implies that $I = R$. Note that we have $I \subseteq R$, so we only have to show the reverse inclusion. Suppose that $a \in R$. Then $a + M \in R/M = J$, so $a + M = b + M$ for some $b \in I$. But then $a - b \in M$, and

$M \subseteq I$, so $a-b \in I$. Since $b \in I$, we conclude $a = (a-b) + b \in I$. Therefore $R = I$.

Thus there are only two possibilities for I: either $I = M$ or $I = R$. Therefore M is maximal.

$\Rightarrow$ Exercise.

Hint: Suppose that J is an ideal in R/M. Show that

$$I = \pi^{-1}(J) = \{a \in R : a + M \in J\}$$

is an ideal in R, and $M \subseteq I \subseteq R$. Since M is maximal, this leaves only two possibilities. $\blacksquare$

<u>Corollary</u>

Let R be a commutative ring with identity, and let M be a proper ideal in R. Then:

$$M \text{ is maximal} \iff R/M \text{ is a field.}$$

<u>Proof:</u>

$\Leftarrow$ Suppose that R/M is a field. Then we know by earlier results that R/M is simple. The preceding theorem therefore implies that M is maximal.

$\Rightarrow$ Suppose that M is maximal. Then the preceding theorem implies that R/M is simple. Thus R/M is a commutative ring with 1 that has no nontrivial ideals. We proved earlier that this implies that R/M is a field. ∎

<u>Example/Exercise</u>

Consider $R = \mathbb{Z}$. Suppose that $n > 0$ is composite, i.e., $n = kl$ with $k, l > 1$.

Exercise: Show $n\mathbb{Z} \subsetneq k\mathbb{Z} \subsetneq \mathbb{Z}$.

Thus $n\mathbb{Z}$ is not a maximal ideal. Therefore $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ is not a field.

Suppose $p > 0$ is prime. Show that $p\mathbb{Z}$ is a maximal ideal, & conclude that $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ is a field.