MATH 4317 (C. Heil) HOMEWORK #1 DUE: May 31, 2007

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

NOTES. Your grade on this and every homework is based on my ability to understand and evaluate what you have written. Therefore it is ESSENTIAL that you COMMUNICATE CLEARLY in your writing. ALL problems require either a proof or an explanation of what you have done, using complete, correct English sentences (mathematical symbols are simply abbreviations for words or phrases and may be used as parts of sentences). I will read EXACTLY what you write and will take what you write literally. I will not fill in missing steps or guess at what you "really mean." Any symbols that you introduce that are not standard must be explained. YOUR EXPLANATIONS NEED NOT BE LENGTHY TO BE CLEAR, but you must carefully and logically demonstrate the validity of your solution to each problem. The words "show," "demonstrate," etc. are all synonyms for "prove." All statements that you make require proof, e.g., if I ask "is statement A true?" then it is not sufficient to answer "yes" or "no", you must explain WHY the answer is yes or no.

1. Define $\mathcal{F} = \{f \;:\; f\colon \mathbf{R} \to \mathbf{R}\}$. Given $f \in \mathcal{F}$ and $a \in \mathbf{R}$, define $\tau_a(f) \in \mathcal{F}$ to be the function whose rule is

$$\tau_a(f)(x) \;=\; f(x+a), \quad x \in \mathbf{R}.$$

Set $\mathbb{F} \;=\; \{\alpha \;:\; \alpha\colon \mathcal{F} \to \mathcal{F}\}$, and define $\varphi\colon \mathbf{R} \to \mathbb{F}$ by $\varphi(a) = \tau_a$. Determine whether φ is injective.

2. Prove that there exists a bijection $f\colon [0,1] \to (0,1)$.

3. Let $\mathbf{N}$ be the set of natural numbers. Define the *power set* $\mathcal{P}(\mathbf{N})$ of $\mathbf{N}$ to be the set of all subsets of $\mathbf{N}$, i.e.,

$$\mathcal{P}(\mathbf{N}) \;=\; \{S : S \subseteq \mathbf{N}\}.$$

Prove that $\mathcal{P}(\mathbf{N})$ is uncountable.

Hint: If you can find an injection of an uncountable set *into* $\mathcal{P}(\mathbf{N})$, then you know that $\mathcal{P}(\mathbf{N})$ is uncountable. So, for example, you might try to show that there exists an injection $f\colon (0,1) \to \mathcal{P}(N)$. To do this, it might be useful to consider the decimal expansion of $x \in (0,1)$: given each $x = 0.d_1d_2d_3\ldots$, you want to associate a *unique* subset of $\mathbf{N}$. (In fact, binary expansions might be simpler to use.)

Alternatively, if you can show that there is a surjection from $\mathcal{P}(\mathbf{N})$ onto an uncountable set, then you know that $\mathcal{P}(\mathbf{N})$ is uncountable. So, you might try to show that there exists a surjection $f\colon \mathcal{P}(\mathbf{N}) \to (0,1)$. Again, decimal or binary expansions might be useful: given any subset S of $\mathbf{N}$, you somehow want to associate a number $x \in (0,1)$ in such a way that you are sure that *every* $x \in (0,1)$ gets hit by this map.

4. Prove that there does not exist a rational number r such that $r^2 = 5$.

5. Let X be a non-empty set and let f and g be defined on X and have bounded ranges in $\mathbf{R}$. Show that

$$\begin{aligned} \inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} &\le \inf\{f(x) + g(x) : x \in X\} \\ &\le \inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}. \end{aligned}$$

Give examples to show that each inequality can be strict.