17. Convergence

Review: There are two basic types of convergence in \( \mathbb{R}^p \):

(a) Norm Convergence

Let \((x_n) = (x_1, x_2, x_3, \ldots)\) be a sequence of vectors in \( \mathbb{R}^p \).

Let \( \|x\| \) be a given norm in \( \mathbb{R}^p \).

Then \((x_n)\) converges in norm to \( y \in \mathbb{R}^p \) if \( \lim_{n \to \infty} \|y - x_n\| = 0 \). That is:

\[
\forall \varepsilon > 0 \exists N > 0 \text{ s.t. } n > N \implies \|y - x_n\| < \varepsilon .
\]

Also write: \( x_n \to y \) or \( y = \lim_{n \to \infty} x_n \).

Remark: All norms on a finite-dimensional space \( \mathbb{R}^p \) are equivalent. That is, if \( \| \cdot \| \) and \( \| \cdot \|' \) are two norms, then

\[
\exists A, B > 0 \text{ s.t. } \forall x \in \mathbb{R}^p, A \|x\| \leq \|x\|' \leq B \|x\| .
\]

Exercise: Therefore \( \lim_{n \to \infty} \|y - x_n\| = 0 \iff \lim_{n \to \infty} \|y - x_n\|' = 0 \).
(b) Componentwise Convergence

Again let \((x_n) = (x_1, x_2, \ldots)\) be a sequence of vectors in \(\mathbb{R}^p\).

Write out the components explicitly:

\[
X_1 = (x_{1,1}, x_{1,2}, \ldots, x_{1,p}) \\
X_2 = (x_{2,1}, x_{2,2}, \ldots, x_{2,p}) \\
\vdots \\
X_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,p}) \\
\vdots
\]

Write the components of \(y\):

\[
y = (y_1, y_2, \ldots, y_p).
\]

Then \((x_n)\) converges componentwise to \(y\) if

\[
\begin{align*}
\lim_{n \to \infty} x_{n,1} &= y_1, \quad \text{1st components of } x_n \text{'s converge to 1st component} \\
\vdots \\
\lim_{n \to \infty} x_{n,p} &= y_p, \quad \text{p'th components of } x_n \text{'s converge to p'th component}
\end{align*}
\]

Ex: \(\mathbb{R}^2\)

\[
X_n = (x_{n,1}, x_{n,2}) \\
y = (y_1, y_2)
\]
**Theorem:** \( x_n \to y \) in norm \( \iff \) \( x_n \to y \) componentwise.

**In \( \mathbb{R}^n \):**

Illustration as digital signals:

\[
\begin{array}{c}
\downarrow x_{n,1} \\
\downarrow x_{n,2} \\
\phantom{\downarrow x_{n,2}} \ldots \\
\downarrow x_{n,n}
\end{array}
\]

Each height converges individually \( \iff \) Areas converge of all.

But this theorem does not generalize to \( \mathbb{R}^n \)-dimensional spaces!
Convergence in $l^1$

$l^1 = \{ x = (x_1, x_2, \ldots) : \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty \}$

Consider a sequence of vectors in $l^1$ (be careful to distinguish sequence index from component index).

Sequence $(x_n) = (x_1, x_2, \ldots)$ in $l^1$ — Now $x_n \in l^1$!

\[ x_1 = (x_{1,1}, x_{1,2}, x_{1,3}, \ldots) = (x_{1,k})_{k=1}^{\infty} \]

\[ \vdots \]

\[ x_n = (x_{n,1}, x_{n,2}, x_{n,3}, \ldots) = (x_{n,k})_{k=1}^{\infty} \]

\[ \vdots \]

Let

\[ y = (y_1, y_2, y_3, \ldots) = (y_k)_{k=1}^{\infty} \]

Then $x_n \to y$ in norm if

\[ \lim_{n \to \infty} \|x_n - y\|_1 = 0 \]

or

\[ \lim_{n \to \infty} \sum_{k=1}^{\infty} |x_{n,k} - y_k| = 0. \]
And \( x_n \to y \) \underline{componentwise\ if} \[
\forall k, \lim_{n \to \infty} x_{n,k} = y_k.
\]
or \[
\forall k, \lim_{n \to \infty} |x_{n,k} - y_k| = 0. \tag{*} (**)\]

Compare \((*)\) & \((***)\)!

Exercise: \( x_n \to y \) in norm \(\implies\) \( x_n \to y \) componentwise.

Example: \(\iff\) is \underline{FALSE}.\ 

Set \[
x_1 = (1, 0, 0, \ldots)
\]
x_2 = (0, 1, 0, \ldots)
x_3 = (0, 0, 1, \ldots)
\vdots
y = (0, 0, 0, \ldots)

Then \(\forall k\) \(x_{n,k} \to 0 = y_k\). \underline{Converges componentwise!}

But \(\|x_n - y\|_1 = 1 \not\to 0!\)

Does \underline{NOT} converge \underline{in norm}!

Remark: Similar for \(l^2\) or \(l^\infty\). \(l^\infty\) norm convergence is called \underline{uniform convergence}.\
Convergence of Functions

Consider a sequence of functions \((f_n) = (f_1, f_2, \ldots)\) on some domain — for simplicity we'll take \(\mathbb{R}\) domain to be \([0, 1]\), but it could just as well be arbitrary. The functions will be real-valued, i.e., \(f_n : [0, 1] \to \mathbb{R}\).

There are many different ways in which the sequence of functions \((f_n)\) might converge to a limit function \(f\). Unlike \(\mathbb{R}^p\), where componentwise convergence & norm convergence are completely equivalent, for functions \(f\) there are many ways — all different & all reasonable — to define convergence.

The analogue of componentwise convergence is pointwise convergence, and additionally there are (infinitely) many different types of norm convergence.
We'll illustrate this with some examples.

Recall: The components of a vector \( \mathbf{x} = (x_1, \ldots, x_p) \) are analogous to function values \( f(x) \) of \( f \).

\[
\begin{array}{c|c|c|c|c}
\text{x}_1 & \text{x}_2 & \text{x}_3 & \ldots & \text{x}_p \\
1 & 2 & 1 & \ldots & p
\end{array}
\]

A vector \( \mathbf{x} = (x_1, \ldots, x_p) \) is really a function \( x : \{1, \ldots, p\} \rightarrow \mathbb{R} \).

Vector = discrete function

Componentwise convergence means convergence of each individual component:

\( X_n = (x_{n1}, \ldots, x_{np}) \)

converges componentwise to

\( Y = (y_1, \ldots, y_p) \) if

\[
\lim_{n \to \infty} x_{nk} = y_k \\
\text{for } k = 1, \ldots, p
\]

Componentwise convergence

Pointwise convergence means convergence of each function value:

\( f_n \) converges pointwise to \( g \) if

\[
\lim_{n \to \infty} f_n(x) = g(x) \\
\text{for each } x \in [0,1].
\]
Example
Let \( f_n \) be the continuous function on \([0,1]\) whose graph is:

\[
\begin{array}{c}
0 & \frac{1}{2} & 1 \\
-1 & \frac{1}{2} & 1 \\
\frac{1}{2} & \frac{1}{2}+rac{1}{n} & 1 \\
\frac{1}{2}-\frac{1}{n} & 1 \\
\end{array}
\]

Exercise: Write down an explicit formula for \( f_n \):

\[
f_n(x) = \begin{cases} 
1, & \frac{1}{2}+\frac{1}{n} \leq x \leq 1 \\
?, & \frac{1}{2}-\frac{1}{n} \leq x \leq \frac{1}{2}+\frac{1}{n} \\
-1, & 0 \leq x \leq \frac{1}{2}-\frac{1}{n} 
\end{cases}
\]

If \( \frac{1}{2} < x \leq 1 \), then eventually \( n \) will be large enough that \( \frac{1}{2}+\frac{1}{n} < x \). From some \( n \) onward, every \( f_n(x) \) will equal 1:

\[
\exists N \text{ s.t. } n>N \Rightarrow f_n(x) = 1
\]

So for any particular \( x \) in the range \( \frac{1}{2} < x \leq 1 \),

\[
\lim_{n \to \infty} f_n(x) = 1.
\]
functions can be discontinuous.

Here is another example of pointwise convergence.

Example

Define $f_n$ to be

$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2n} \\ \text{?}, & \frac{1}{2n} < x \leq \frac{1}{n} \\ \text{?}, & \frac{1}{n} \leq x \leq 1 \\ 0, & \frac{1}{n} \leq x \leq 1 \end{cases}$$

If we choose any $0 < x < 1$, then for all $n$ large enough, $0 < \frac{1}{n} < x$, so

$$\exists N \text{ s.t. } n > N \Rightarrow f_n(x) = 0.$$  

Thus for all $x > 0$ we have $\lim_{n \to \infty} f_n(x) = 0$. And for $x = 0$ we have $f_n(0) = 0 \forall n$, so $\lim_{n \to \infty} f_n(0) = 0$.

Therefore $\lim_{n \to \infty} f_n(x) = 0 \forall x \in [0,1]$, so
Likewise, show that

\[ \lim_{n \to \infty} f_n(x) = -1. \]

For \( 0 \leq x < \frac{1}{2} \) we have \( f_n(x) \) and for every \( n \), so

\[ \lim_{n \to \infty} f_n(\frac{1}{2}) = 0. \]

And for \( x = \frac{1}{2} \), \( f_n(\frac{1}{2}) = 0 \) for every \( n \), so

Thus for each \( x \in [0, 1] \), \( f_n(x) \) converges as \( n \to \infty \), \( x \)

\[ \lim_{n \to \infty} f_n(x) = g(x) = \begin{cases} 
1, & \frac{1}{2} < x \leq 1 \\
0, & x = \frac{1}{2} \\
-1, & 0 \leq x < \frac{1}{2} 
\end{cases} \]

For each \( x \), \( f_n(x) \) converges to \( g(x) \)

This is pointwise convergence:

\( f_n \) converges pointwise to \( g \)

Sometimes for emphasis we will write that
\n\[ f_n(x) \] converges pointwise to \( g(x) \) for each \( x \).

Note that \( g \) is a pointwise limit of continuous
Exercise

Show that if $f_n$ grows taller with $n$:

Then we still have that $f_n$ converges pointwise to zero.
Norm Convergence: $L^1, L^2, \&$ uniform norms

Pointwise convergence is just one type of convergence of functions. Another is norm convergence. But there isn't just one type of norm convergence—there are infinitely many different norms.

The idea of norm convergence is that we find a norm that somehow measures the "distance" between two functions $f \& g$. We think of $f \& g$ as being two points in a vector space $V$, and find some way of measuring the lengths of vectors in $V$:

The distance between $f \& g$ is $\|f - g\|$, determined
according to whatever $2$ formula it for that particular norm is.

Convergence with respect to $2$ norm just means that $2$ distance between $f_n$ & $g$ shrinks to zero as $n \to \infty$.

\[
\begin{array}{c}
\text{.} \\
\text{.} \\
\text{.} \\
\text{.}
\end{array}
\]

\[
\begin{array}{c}
\text{.} \\
\text{.} \\
\text{.} \\
\text{.}
\end{array}
\]

\[
\text{.} \\
\text{.} \\
\text{.} \\
\text{.}
\]

\[
\begin{array}{c}
\text{.} \\
\text{.} \\
\text{.} \\
\text{.}
\end{array}
\]

The distance between $f_n$ & $g$ is $\|g-f\|$; but its definition is given by the rule for $2$ norm, it's not a physical distance as the Euclidean distance between points in $\mathbb{R}^n$ is.

The norms we most often use are not so strange -- they're really just continuous versions of $\ell^p$-norms on $\mathbb{R}^n$. 
$L^1$-norm convergence

The $L^1$-norm for functions is a generalization of the $L^1$-norm for vectors in $\mathbb{R}^n$, or the $L^1$-norm for infinite sequences.

Contrast a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, which is a discrete function, with a function $f$ on a domain $D \subseteq \mathbb{R}$ (e.g., $D = [0, 1]$ or $\mathbb{R}$).

\[ \text{area of each box } \square \text{ is } |x_k| \]

\[ \text{total area is } \|x\|_1 = \sum_{k=1}^{n} |x_k| = |x_1| + \ldots + |x_n| \]

\[ \text{total area is } \|f\|_1 = \int_{a}^{b} |f(x)| \, dx \]
Definition
Let $D \subseteq \mathbb{R}$ be given. The $L^1$-norm of a function $f: D \to \mathbb{R}$ is

$$\|f\|_1 = \int_D |f(x)| \, dx.$$ 

Technical Problem
The meaning of the integral for completely arbitrary functions is subtle, and is covered in a course on measure theory and integration. In fact, there exist "nonmeasurable" functions that cannot be integrated.

To avoid complications, we shall only apply the integral to functions that are piecewise continuous:

\[ \quad \]

- a piecewise continuous function

In this case, the integral is just the ordinary Riemann integral we learned in Calculus.

For more general functions, the Lebesgue integral extends the Riemann integral to much more general functions.
The distance between two functions \( f \) & \( g \), measured using the \( L^1 \)-norm, is:

\[
\| f - g \|_1 = \int |f(x) - g(x)| \, dx
\]

= area between \( f \) & \( g \) graphs.

Example:
If \( f, g \) look like this:

\[ f \]
\[ g \]

Then the distance \( \| f - g \|_1 \) can be very small even though there's an \( x \) where \( f(x) \) & \( g(x) \) are far apart — it just depends on the total area between \( f \) & \( g \).
Example

The function $g - f_n$ looks like this:

Therefore, $\|g - f_n\|_1 = \int_0^1 |g(x) - f_n(x)| \, dx$

is the area of the shaded region above. The distance between $f_n$ and $g$ is the sum of the areas of the two triangles.
\[ \|g - f_n\|_1 = \int_0^1 |g(x) - f_n(x)| \, dx \]
\[ = \frac{1}{2} \text{ base} \times \text{ height} + \frac{1}{2} \text{ base} \times \text{ height} \]
\[ = \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n} \cdot 1 \]
\[ = \frac{1}{n} \]

These two functions are a distance \( \frac{1}{n} \) apart, according to the \( L^1 \)-norm definition of distance.

Since the distance from \( f_n \) to \( g \) goes to zero:
\[ \lim_{n \to \infty} \|g - f_n\|_1 = \lim_{n \to \infty} \frac{1}{n} = 0 \]
we say that \( f_n \) converges to \( g \) in \( L^1 \)-norm.

**Definition**
A sequence of piecewise continuous functions \( f_n \) converges to a piecewise continuous function \( g \) in \( L^1 \)-norm if
\[ \lim_{n \to \infty} \|g - f_n\|_1 = 0. \]

We write \( f_n \to g \) in \( L^1 \)-norm.
Example

\[ f_n \]

If \( f_n \) is as above and \( g = 0 \), then

\[ \| g - f_n \|_1 = \int_0^1 |g(x) - f_n(x)| \, dx \]

\[ = \int_0^1 |f_n(x)| \, dx \]

\[ = \frac{1}{2} \cdot \frac{1}{n} \cdot 1 \]

\[ = \frac{1}{2n} \]

\[ \to 0 \quad \text{as} \quad n \to \infty, \]

so \( f_n \to 0 \) in \( L^1 \)-norm.

Remember that an earlier example showed that \( f_n \to 0 \) pointwise, but these are two different types of convergence! Compare this with the next example.
Example

\[ f_n \]

A previous example showed that if \( f_n \) is as above, then \( f_n \to 0 \) pointwise. However, the \( L^1 \)-distance from \( f_n \) to 0 is

\[
\|0 - f_n\|_1 = \int_0^1 |0 - f_n(x)| \, dx
\]
\[
= \int_0^1 |f_n(x)| \, dx
\]
\[
= \frac{1}{2} \cdot \frac{1}{n} \cdot n
\]
\[
= \frac{1}{2} \quad \text{for all } n.
\]

Each \( f_n \) is \( \frac{1}{2} \) unit away from 0, according to the \( L^1 \)-norm!
So $f_n$ never gets close to 0, according to the $L^1$-norm! $f_n \to 0$ in $L^1$-norm, even though $f_n$ converges pointwise to 0.

In general:

$$f_n \to f \text{ in } L^1\text{-norm} \not\implies f_n \to f \text{ pointwise}$$

Neither type of convergence implies the other!

We've seen an example (the preceding one) of functions $f_n$ s.t. $f_n \to 0$ pointwise but $f_n \not\to 0$ in $L^1$-norm. Now we give an example of function $f_n$ s.t. $f_n \to 0$ in $L^1$-norm but $f_n \not\to 0$ pointwise.
Example: The "Marching Boxes" example.

Consider $f_1, f_2, f_3, \ldots$ like this:

\[ f_1 \]

\[ f_2 \]

\[ f_3 \]

\[ f_4 \]

\[ f_5 \]

\[ f_6 \]

\[ f_7 \]

etc.
Then $f_n \to 0$ in $L^1$-norm, because

$$\|0 - f_1\|_1 = \|f_1\|_1 = \int_0^1 |f_1(x)| \, dx = 1$$

$$\|0 - f_2\|_1 = \|f_2\|_1 = \frac{1}{2}$$

$$\|0 - f_3\|_1 = \|f_3\|_1 = \frac{1}{3}$$

$$\|0 - f_4\|_1 = \|f_4\|_1 = \frac{1}{3}$$

$$\|0 - f_5\|_1 = \|f_5\|_1 = \frac{1}{5}$$

$$\|0 - f_6\|_1 = \|f_6\|_1 = \frac{1}{5}$$

$\frac{1}{5}$

$\frac{1}{5}$

$\frac{1}{5}$

$\frac{1}{5}$

etc.

$$\|0 - f_{n+1}\|_1 = \|f_{n+1}\|_1 \to 0 \text{ as } n \to \infty,$$

so $f_n \to 0$ in $L^1$-norm.

But $f_n$ does NOT converge to 0 for
anything else) pointwise!! Why??

$L^2$-norm

The $L^2$-norm for functions generalizes the Euclidean norm for vectors in $\mathbb{R}^n$. The $L^2$-norm of a piecewise continuous function $f$ on a domain $D$ is

$$\|f\|_2 = \left( \int_D |f(x)|^2 \, dx \right)^{1/2}.$$  

We say that $f_n \to f$ in $L^2$-norm if

$$\lim_{n \to \infty} \|g-f_n\|_2 = \lim_{n \to \infty} \left( \int_D |g(x)-f_n(x)|^2 \, dx \right)^{1/2} = 0.$$  

Exercise: Recheck the preceding example to see if we have $L^2$-norm convergence.
Note.

On a finite domain, such as \([0,1]\), there is a relation between \(L^1\) & \(L^2\) norms. Suppose

\[ f: [0,1] \to \mathbb{R} \] is piecewise continuous. Then

\[ \|f\|_1 = \int_0^1 |f(x)| \, dx \]

\[ = \int_0^1 |f(x)| \cdot 1 \, dx \]

\[ = \langle |f|, 1 \rangle \quad \text{inner product of } |f| \text{ & } 1 \]

\[ \leq \|f\|_2 \|1\|_2 \quad \text{Cauchy-Schwarz inequality} \]

\[ = (\int_0^1 |f(x)|^2 \, dx)^{\frac{1}{2}} \left( \int_0^1 1^2 \, dx \right)^{\frac{1}{2}} \]

\[ = \|f\|_2 \cdot 1 \]

\[ = \|f\|_2. \]

In particular,

\[ \|g - f_n\|_1 \leq \|g - f_n\|_2, \]

for functions on the domain \([0,1]\).
Exercise: Show that implies that
\[ f_n \to g \text{ in } L^2\text{-norm} \implies f_n \to g \text{ in } L^1\text{-norm}. \]

However, \( f_n \to g \text{ in } L^1\text{-norm} \not\implies f_n \to g \text{ in } L^2\text{-norm} \) (in general). Try

\[ f_n(x) = \begin{cases} \frac{1}{n} x^{-\frac{1}{2}}, & 0 < x \leq 1, \\ 0, & x = 0. \end{cases} \]

Then \( f_n \to 0 \text{ in } L^1\text{-norm} \) but \( f_n \not\to 0 \text{ in } L^2\text{-norm} \).

Further, on an infinite domain, \( \overline{d} \) is no relation between \( L^1 \) & \( L^2 \) norm in general.

For example, on the domain \((1, \infty)\), consider

\[ f_n(x) = \frac{1}{x}, \quad x \geq 1. \]
$L^\infty$-norm = Uniform Norm

The $L^\infty$-norm for piecewise continuous functions is also called \textit{uniform norm}, \& is defined by

$$\|f\|_{L^\infty} = \sup_{x \in D} |f(x)| \quad (\star)$$

where $D$ is the domain of $f$.

Note: For completely arbitrary functions, there is a subtle difference between the $L^\infty$-norm and the uniform norm or sup-norm given by $(\star)$. That distinction is covered in a course on measure theory.

We say that

$$f_n \to g \text{ in } L^\infty\text{-norm or uniformly if}$$

$$\lim_{n \to \infty} \|g - f_n\|_{L^\infty} = \lim_{n \to \infty} \left( \sup_{x \in D} |g(x) - f_n(x)| \right) = 0.$$ 

This is the "maximum deviation" between $f$ \& $g$. 
The "maximum deviation" between $f$ and $g$ is $\|f-g\|_{\infty}$.

These two functions are far apart in $L^\infty$-norm, even though $f(x)$ & $g(x)$ are close for most $x$'s, there's one $x$ where $|f(x) - g(x)|$ is large.

$\|f-g\|_{\infty}$ is large.

On the other hand, the area between the two graphs is small, so

$\|f-g\|_1 = \int |f(x) - g(x)| \, dx$ is small.
Example 1:

We saw before that \( f(x) \) converges pointwise to the zero function:

\[
\forall x \in [0, 1], \quad \lim_{n \to \infty} |f_n(x) - 0| = 0
\]

For each individual \( x \), \( f_n(x) \) eventually becomes zero.

However, there is always some \( x_n \) (depending on \( n \!\)!) where \( f_n(x_n) \) is zero and \( 0 \) are for apart.

The \( L^\infty \) distance between \( f_n \) and the zero function is

\[
\|f_n - 0\|_\infty = \sup_{x \in [0, 1]} |f_n(x) - 0| = 1
\]
Thus $\|f_n - 0\|_\infty = 1$ $\forall n$

So $f_n \not\to 0$ in $L^\infty$-norm

or $f_n \not\to 0$ uniformly!

Thus:

Pointwise convergence $\not\to$ uniform convergence

Exercise Show that

uniform convergence $\implies$ pointwise convergence

Exercise Show that on a finite domain $D$,

uniform convergence $\implies L^1$ convergence.

But show that on an infinite domain (such as $D = \mathbb{R}$),

uniform convergence $\not\to L^1$ convergence.

Hint: $f_n(x) = \frac{1}{n}$
Example/Exercise: Show

\[ L^1 \text{ convergence } \not\Rightarrow \text{ uniform convergence} \]

Consider

\[
\begin{array}{c}
\sum_{n} f_n \\
\frac{1}{n}
\end{array}
\]

Picture for $L^\infty$-norm

\[ \| f - g \|_\infty = r \implies \sup_x |f(x) - g(x)| = r \]

\[ \implies |f(x) - g(x)| \leq r \quad \forall x \]

If \[ \| f - g \|_\infty < r \], then \[ g \] must always take values between the dotted lines.
So: \[ g \in B_r(f) \iff \| f - g \|_\infty < r \]

Any function \( g \) that lies between the dotted lines belongs to the ball of radius \( r \) centered at \( f \) (with respect to \( \| \cdot \|_\infty \)-norm!)

**Bartle Notation**

Instead of just considering functions \( f \) that map numbers to numbers, we can consider functions that map vectors to vectors.

If \( f: \mathbb{R}^p \to \mathbb{R}^q \), the analog of \( \| \cdot \|_\infty \)-norm is

\[ \| f \| = \sup_{x \in \mathbb{R}^p} \| f(x) \| \]

(norm of \( f(x) \) \( \in \mathbb{R}^q \))

(or could have \( f \) defined on a domain \( I \subseteq \mathbb{R}^p \) instead of all of \( \mathbb{R}^p \)).
Barthe defines

$$B_{pq}(I) = \{ f : I \rightarrow \mathbb{R}^q : \|f\| = \sup_{x \in I} \|f(x)\| < \infty \}$$

Gibbs' Phenomenon

Set

$$f_n(x) = \frac{4}{\pi^3} \sum_{k=1}^{\infty} \frac{\sin \left( (2k-1) \frac{2\pi x}{2k-1} \right)}{2k-1}$$

Then

$$g(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ 0 & x = 0, \frac{1}{2}, 1 \\ -1 & \frac{1}{2} < x < 1 \end{cases}$$

$$f_n$$ converges to $$g$$ \textbf{pointwise} and in $$L^1$$-norm but not uniformly!

See figures showing $$f_1, f_2, f_3, f_{10}, f_{50}$$
What does it mean for a sequence of functions \( (f_n) \) to converge to a limit function \( g \)?

**Pointwise Convergence**

For each individual \( x \),

\[ \lim_{n \to \infty} f_n(x) = g(x) \]

Book calls this "convergence".

**Norm Convergence**

All \( x \)’s are somehow involved

\[ \lim_{n \to \infty} \| f_n - g \| = 0 \]

- \( L^1 \)
- \( L^2 \)
- \( L^\infty \)

\( \| \) uniform convergence
Comparison

Pointwise Convergence

\[ \forall x, \quad f_n(x) \to f(x). \quad \text{Means:} \quad \forall x, \quad \forall \varepsilon > 0 \quad \exists N \quad \forall n \geq N \implies |f_n(x) - f(x)| < \varepsilon \]

\[ \uparrow \quad \uparrow \]

N depends on x (\& \varepsilon)

For a given \( \varepsilon \), each \( x \) may have a different \( N \)

Uniform Convergence

\[ f_n \to f \quad \text{uniformly means:} \quad \|f_n - f\|_{\infty} \to 0 \]

\[ \sup_{x} |f_n(x) - f(x)| \to 0 \]

\[ \forall \varepsilon > 0 \quad \exists N \quad \forall x, \quad \forall n \geq N \implies |f_n(x) - f(x)| < \varepsilon \]

\[ \forall \varepsilon > 0 \quad \exists N \quad \forall x, \quad \forall n \geq N \implies \int |f_n(x) - f(x)| \, dx < \varepsilon \]

Given \( \varepsilon \), one \( N \) works for all \( x \) simultaneously "uniform"

\( L^1 \) convergence

\[ f_n \to f \quad \text{in} \ L^1 \quad \text{norm means:} \quad \|f_n - f\|_{L^1} \to 0 \]

\[ \forall \varepsilon > 0 \quad \exists N, \quad \forall n \geq N \implies \int |f_n(x) - f(x)| \, dx < \varepsilon \]

doesn't require \[ |f_n(x) - f(x)| < \varepsilon \ \forall x \]
For $L^1$ or $L^\infty$: Convergence $\Rightarrow$ Cauchy.
Same proof, true for any norm.

Problem: Does Cauchy $\Rightarrow$ Convergent? Check uniform norm case.

Yes. Proof:

Assume $(f_n)$ is Cauchy in $L^\infty(\mathbb{R})$-norm (uniform norm).

Choose $\epsilon > 0$.\exists N > 0 \text{ s.t. } n > N \Rightarrow \|f_n - f_m\|_\infty < \epsilon.

or $\sup_x |f_n(x) - f_m(x)| < \epsilon$.

Hence $m, n > N \Rightarrow |f_n(x) - f_m(x)| < \sup_x |f_n(x) - f_m(x)| < \epsilon.$ $\forall x.$

Thus for each individual $x$, $(f_n(x))$ is Cauchy. Sequence of numbers.

Define $f(x) = \lim_{n \to \infty} f_n(x)$.

We have then that $f_n \to f$ Pointwise.

Claim: $f_n \to f$ uniformly.

To see this, let $x$ be fixed. Then we know $(f_n)$ is Cauchy in $L^\infty$, so

$a_n = \|f_m(x) - f_n(x)\|_\infty < \epsilon$ for all $m, n > N$. 
Hence  \( \lim_{n \to \infty} |f_m(x) - f(x)| = \lim_{n \to \infty} a_n \leq \varepsilon \).

But \( f_m(x) \) is fixed.

\[
\lim_{n \to \infty} |f_m(x) - f(x)| = \lim_{n \to \infty} |f_m(x) - f_n(x)| \\
\leq \varepsilon \\
= |f_m(x) - \lim_{n \to \infty} f_n(x)| \\
= |f_m(x) - f(x)|
\]

So  \( \uparrow \leq \varepsilon \).

Thus \( \forall x, |f_m(x) - f(x)| \leq \varepsilon \).

So  \( \sup_{x \in \mathbb{R}} |f_m(x) - f(x)| \leq \varepsilon \).

So  \( \|f_m - f\|_{\infty} \leq \varepsilon \) \( \forall m \geq M \).

Thus \( f_m \to f \) in \( L^\infty \) norm (uniformly).