

23. Uniform Continuity

In an earlier theorem, we showed that f is continuous at all points in its domain if:

$$\forall a \in D(f), \forall \varepsilon > 0, \exists \delta > 0 \text{ such that}$$

$$x \in D(f) \ \& \ \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

The value of δ can depend on both a & ε .

If δ doesn't depend on a , we say that f is uniformly continuous. Precisely:

Definition

f is uniformly continuous on $A \subseteq D(f)$ if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that}$$

$$\forall a, x \in A, \quad \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

Main point: Given ε , there is a single δ such that

$$\|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon \text{ for } \underline{\text{all}} \ x, a !$$

δ is independent of a !

Exercise: $f(x) = \frac{1}{x}$ is NOT uniformly continuous on $(0, \infty)$.

It is uniformly continuous on $[c, \infty)$ if $c > 0$.

Theorem

If f is continuous & $K \subseteq D(f)$ is compact, then f is uniformly continuous on K .

Proof:

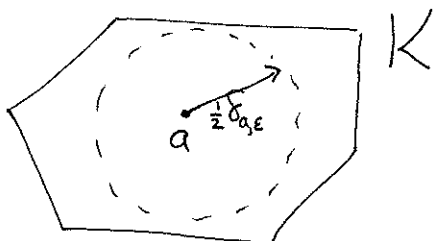
Since f is continuous on K , for each $a \in K$ & each $\varepsilon > 0$ there exists a number $\delta_{a,\varepsilon}$ such that

$$x \in K \text{ \& } \|x - a\| < \delta_{a,\varepsilon} \implies \|f(x) - f(a)\| < \frac{\varepsilon}{2}$$

Then ^{for a given ε} the collection of open balls

$$\left\{ B_{\frac{\delta_{a,\varepsilon}}{2}}(a) : a \in K \right\}$$

covers all of K (why?)



(3)

Since K is compact, it must be covered by finitely many of these balls, say the ones centered at the points a_1, \dots, a_N . That is,

$$K \subseteq B_{\delta_{a_1, \epsilon}}(a_1) \cup \dots \cup B_{\delta_{a_N, \epsilon}}(a_N). \quad (*)$$

Define

$$\delta_\epsilon = ~~\epsilon~~ \frac{1}{2} \min \{ \delta_{a_1, \epsilon}, \dots, \delta_{a_N, \epsilon} \}.$$

We will show that

$$\forall a, x \in K, \quad \|x - a\| < \delta_\epsilon \implies \|f(x) - f(a)\| < \epsilon.$$

This is exactly what we have to show to prove

that f is uniformly continuous on K .

So, to do this, suppose $a, x \in K$ ~~are such that~~

~~are such that~~ $\|x - a\| < \delta_\epsilon$.

From (*), we must have

$$a \in B_{\frac{1}{2}\delta_{a_i, \epsilon}}(a_i) \quad \text{for some } i = 1, \dots, N.$$

This means that

$$\|a - a_i\| < \frac{1}{2} \delta_{a_i, \varepsilon} < \delta_{a_i, \varepsilon}.$$

But then

$$\begin{aligned} \|x - a_i\| &= \|x - a + a - a_i\| \\ &\leq \|x - a\| + \|a - a_i\| \\ &< \delta_\varepsilon + \frac{1}{2} \delta_{a_i, \varepsilon} \\ &\leq \frac{1}{2} \delta_{a_i, \varepsilon} + \frac{1}{2} \delta_{a_i, \varepsilon} \\ &= \delta_{a_i, \varepsilon}. \end{aligned}$$

Therefore

$$\|f(a) - f(a_i)\| < \frac{\varepsilon}{2} \quad \& \quad \|f(x) - f(a_i)\| < \frac{\varepsilon}{2},$$

so

$$\begin{aligned} \|f(a) - f(x)\| &= \|f(a) - f(a_i) + f(a_i) - f(x)\| \\ &\leq \|f(a) - f(a_i)\| + \|f(a_i) - f(x)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \quad \square \end{aligned}$$

Contractions

A function f is Lipschitz if $\exists K > 0$ s.t.

$$\forall x, y \in D(f), \quad \|f(x) - f(y)\| \leq K \|x - y\|.$$

~~works~~ Note K is not unique: if one value of K works, then so will any $K' \geq K$.

If we can take $K < 1$ then f is called a contraction.

Fixed Point Theorem

Let $f: \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a contraction. (Note $D(f) = \mathbb{R}^p$, $R(f) \subseteq \mathbb{R}^p$).

Then f has a unique fixed point.

That is, \exists unique $u \in \mathbb{R}^p$ s.t. $f(u) = u$.

Proof:

By def., $\exists K < 1$ s.t.

$$\forall x, y \in \mathbb{R}^p, \quad \|f(x) - f(y)\| \leq K \|x - y\|.$$

Let x_i be any particular point in \mathbb{R}^p .

Define

$$x_2 = f(x_1)$$

$$x_3 = f(x_2)$$

$$x_4 = f(x_3)$$

⋮

We'll show that (x_n) converges in \mathbb{R}^p , & that its limit is the unique fixed point of f .

Note:

$$\|x_3 - x_2\| = \|f(x_2) - f(x_1)\| \leq K \|x_2 - x_1\|.$$

$$\|x_4 - x_3\| = \|f(x_3) - f(x_2)\|$$

$$\leq K \|x_3 - x_2\| \leq K^2 \|x_2 - x_1\|$$

etc. By induction,

$$\|x_{n+1} - x_n\| \leq K^{n-1} \|x_2 - x_1\|.$$

Hence if $m > n$ then

$$\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \dots + \|x_{n+1} - x_n\|$$

$$\leq (K^{m-2} + K^{m-3} + \dots + K^{n-1}) \|x_2 - x_1\|$$

$$\leq \frac{K^{n-1}}{1-K} \|x_2 - x_1\|.$$

Exercise: It follows from the fact (x_n) is a Cauchy sequence, hence converges. Let $u = \lim_{n \rightarrow \infty} x_n$.

Then since f is continuous,

$$f(u) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = u.$$

Thus u is a fixed point of f .

Suppose v was also a fixed point, i.e., $f(v) = v$.

Then

$$\|u - v\| = \|f(u) - f(v)\| \leq K \|u - v\|.$$

If $u \neq v$ then we could divide by $\|u - v\|$ to get $1 \leq K$,

which is a contradiction. Therefore we must have $u = v$,

so the fixed point is unique. \square