1.3 Measures

Recall our motivation from Section 1.1: we were hoping to assign to every set $E \subseteq \mathbb{R}$ a real number $\mu(E)$ in such a way that

\begin{enumerate}
  \item $0 \leq \mu(E) \leq \infty$
  \item $E_1, E_2, \ldots$ disjoint $\Rightarrow \mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$
  \item $\mu(E + h) = \mu(E)$ for $E + h = \{x + h : x \in E\}$
  \item $\mu([-1, 1]) = 1$.
\end{enumerate}

We showed that it is impossible to construct such a function $\mu$.

But perhaps we can relax some of the requirements?

For example, we will construct the Lebesgue measure which satisfies 1-4 except not for every subset of $\mathbb{R}$. Instead, they only hold for $E$ in an appropriate $\sigma$-algebra of subsets of $\mathbb{R}$, the Lebesgue $\sigma$-algebra.

This is perhaps the premier example of a measure $\mu$ is $\mathbb{R}$ one to keep in mind for building intuition, but there are other...
possible ways of relaxing requirements a-d.

Example
Lebesgue measure is essentially predicated on the idea that if \([a, b]\) is an interval in \(\mathbb{R}\), then its measure should be its length, i.e.,

\[
\mu([a, b]) = b - a,
\]

and then somehow we extend from just intervals to more general subsets of \(\mathbb{R}\).

But there might be other natural "sizes" to associate to sets. For example, instead of asking how long a set is, it might be more important to us to ask where the set is. The sets that are "important" to us might be \(\mathbb{R}\) sets that contain the origin. Let's define

\[
\delta(E) = \begin{cases} 1, & \text{if } 0 \in E \\ 0, & \text{if } 0 \notin E. \end{cases}
\]

Then we have:

a. \(0 \leq \delta(E) < \infty\) \(\forall E \subseteq \mathbb{R}\),

b. \(E_1, E_2, \ldots \) disjoint \(\Rightarrow \mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)\)
However:

\[ \delta(E+h) \neq \delta(E) \text{ in general} \]

\[ \delta([0,1]) = 1 \text{ but } f([a,b]) \neq b-a \text{ in general} \]

Still, this may be an entirely reasonable way to "measure" subsets of \( \mathbb{R} \) — and furthermore we can measure every subset of \( \mathbb{R} \) this way. The function \( \delta : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \) is called the \underline{\( \delta \)-measure} on \( \mathbb{R} \).

More generally, the \underline{\( \delta \)-measure}, Dirac measure, or \underline{point mass} at \( a \in \mathbb{R} \) is

\[
\delta_a(E) = \begin{cases} 1, & a \in E \\ 0, & a \notin E \end{cases}
\]

Now, what are \( \delta \) important properties \( \delta \) we will require every measure to have? These are given in the next definitions.
First, what will we measure?

**Definition**

If $X$ is a set and $M$ is a σ-algebra of subsets of $X$, then we call $(X, M)$ a measurable space.

The sets in $M$ are called measurable subsets of $X$.

So, we fix a space $X$ and class of subsets of $X$ that we would like to measure. Now, what do we require of this "measure"?

**Definition**

A measure $\mu$ on a measurable space $(X, M)$ is a function $\mu : M \rightarrow [0, \infty]$ that satisfies:

1. $\mu(\emptyset) = 0$.
2. Countable Additivity: If $E_1, E_2, \ldots \in M$ are disjoint, then

   \[
   \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)
   \]

Note that all terms in this series are nonnegative, so the series either converges to a finite value or is $\infty$. 
Note:
If we have finitely many disjoint sets $E_1, \ldots, E_n \in \mathcal{M}$, then we can set $E_k = \emptyset$ for $k > n$ & obtain a disjoint collection $E_1, E_2, \ldots$. Countable subadditivity therefore implies
\[
\mu(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i).
\]
Thus
\[
\text{countable additivity} \implies \text{finite additivity}.
\]
However, the converse fails in general.

Example:
The $\delta$-measure defined before is a measure on $(\mathbb{R}, \mathcal{M})$ for any $\sigma$-algebra $\mathcal{M}$ on $\mathbb{R}$. 
Exercise: Counting Measure

Here is another measure that is "extreme" in a different way than $\delta$.
Let $X$ be any set, $M = \mathcal{P}(X)$, and define

$$\mu(E) = \begin{cases} 
\# \text{elements of } E & \text{if } E \text{ is finite} \\
\infty & \text{if } E \text{ contains infinitely many elements.}
\end{cases}$$

Show that $\mu$ is a measure on $X$.

So, for example, the Lebesgue measure (once we define it) of an interval $(a,b)$ will be $b-a$, the $\delta$ measure of $(a,b)$ is either 0 or 1, and the counting measure of $(a,b)$ is $\infty$ if $a < b$.

Exercise
Let $M = \mathcal{P}(X)$. Show that

$$\mu(E) = \begin{cases} 
0, & \text{if } E \text{ is finite} \\
\infty, & \text{if } E \text{ is infinite}
\end{cases}$$

is finitely additive but not countably additive.
Definition:

If \( \mu \) is a measure on a measurable space \((X, M)\), then we say \( (X, M, \mu) \) is a measure space.

Special Terminology:

a. If \( \mu(X) < \infty \) then we say \( \mu \) is a finite measure.

b. If \( \exists \) countably many sets \( E_1, E_2, \ldots \in M \) s.t.

\[
\mu(E_i) < \infty \quad \forall i \quad \text{and} \quad \bigcup_{i=1}^{\infty} E_i = X
\]

then we say \( \mu \) is \( \sigma \)-finite.

c. If every \( E \in M \) with \( \mu(E) = \infty \) has a

subset \( F \subseteq E \) s.t. \( F \in M \) & \( 0 < \mu(F) < \infty \),

then we say that \( \mu \) is semi-finite.
Examples

a. Let $\delta$ be the $\delta$-measure on $\mathbb{R}$. Then

$$\delta(\mathbb{R}) = 1 < \infty$$

since $0 \in \mathbb{R}$, so $\delta$ is a finite measure.

b. Suppose that $\mu$ is Lebesgue measure on $\mathbb{R}$ (which we have not defined precisely yet). Then $\mu(\mathbb{R}) = \infty$, but we can write

$$\mathbb{R} = \bigcup_{k \in \mathbb{Z}} [k, k+1]$$

and $\mu([k, k+1]) = 1 < \infty$ for.

Hence Lebesgue measure is $\sigma$-finite, but not finite.

c. Let $\mu$ be counting measure on $\mathbb{R}$. Then $\mu(\mathbb{R}) = \infty$

so $\mu$ is not finite. Further, $\mu$ is not $\sigma$-finite because only finite subsets of $\mathbb{R}$ have finite measure, & we cannot write $\mathbb{R}$ as a countable
union of finite sets. On the other hand, $\mu$ is semi-finite, since every subset of $\mathbb{R}$ has a non-empty finite subset.

**Exercise:**

Finite $\Rightarrow$ $\sigma$-finite $\Rightarrow$ semi-finite

The examples above show that the converse implications do not hold in general.

**Abbreviations on $\mathbb{R}$**

From now on, we will use the shorthand notations

- $\mu(a,b) = \mu((a,b))$
- $\mu([a,b)) = \mu([a,b))$
- $\mu(x) = \mu(\{x\})$

etc.
Basic Properties of Measures: Monotonicity

Let \((X, \mathcal{M}, \mu)\) be a fixed measure space.

Monotonicity

If \(E, F \in \mathcal{M}\) and \(E \subseteq F\) then \(\mu(E) \leq \mu(F)\).

Proof:
Since \(E, F \in \mathcal{M}\), we also have \(F \setminus E = F \cap E^c \in \mathcal{M}\).

Further, \(E \setminus F\) are disjoint, so by additivity we have

\[
\mu(F) = \mu(E \cup (F \setminus E))
\]

\[
= \mu(E) + \mu(F \setminus E)
\]

\[
\geq \mu(E).
\]

Remark:
Later we will study signed measures, which satisfy countable additivity but allow \(-\infty \leq \mu(E) \leq +\infty\). Unfortunately, signed measures need not be monotonic! In particular, the following corollary does not hold for signed measures.
Corollary

Suppose \( E \in M \) & \( \mu(E) = 0 \). If \( F \in M \) & \( F \subseteq E \), then \( \mu(F) = 0 \) as well.

Thus, all measurable subsets of a zero measure set have zero measure.

Zero measure sets will play an important role in measure theory. In general, if \( E \in M \) & \( \mu(E) = 0 \), only some of the subsets \( F \subseteq E \) need belong to \( M \). If it so happens that all subsets of a zero measure set are measurable, then we say the measure is complete.

Definition

If \( \mu \) has the property that

\[
E \in M, \ \mu(E) = 0 \Rightarrow F \in M \text{ for all } F \subseteq E,
\]

then we say that \( \mu \) is complete.

We'll return to this issue of completeness later.

Beware: The term "complete" is heavily overused & appears in many completely unrelated contexts.
Remark: Earlier, during the proof of monotonicity, we showed that

\[ E, F \in M, \ E \subseteq F \implies \mu(F) = \mu(E) + \mu(F \setminus E). \]

We are very tempted to continue as by writing

\[ \mu(F \setminus E) = \mu(F) - \mu(E) \]

But does this make any sense? $\infty - \infty$ is undefined. On the other hand, if $\mu(E) < \infty$
then both sides of this equation do exist.

**Exercise.** Show that if $E, F \in M, \ E \subseteq F, \ \& \ \mu(E) < \infty$ then

\[ \mu(F \setminus E) = \mu(F) - \mu(E) \]

holds. A statement like this means that if one side is finite then so is the other and they are equal, & if one side is $\infty$ then so is the other side.

Moral: Never write $\infty - \infty$. 
Basic Properties of Measures: Subadditivity

In the definition of a measure we have a property of countable additivity: if \( E_1, E_2, \ldots \in \mathcal{M} \) are disjoint, then

\[
\mu \left( \bigcup E_i \right) = \sum \mu(E_i) \quad \text{disjoint sets.}
\]

If the \( E_i \) are not disjoint then computing \( \sum \mu(E_i) \) should count some parts of \( \bigcup E_i \) more than once. So in this case we expect an inequality between \( \mu(\bigcup E_i) \) and \( \sum \mu(E_i) \).

Countable Subadditivity

If \( E_1, E_2, \ldots \in \mathcal{M} \), then \( \mu \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \mu(E_i) \).

Proof:

Use the disjointization trick: set

\[
F_1 = E_1, \quad F_2 = E_2 \setminus E_1, \quad F_3 = E_3 \setminus (E_1 \cup E_2), \ldots
\]

Then \( F_1, F_2, \ldots \in \mathcal{M} \) and they are disjoint. Further, \( F_i \subseteq E_i \).

So by countably additivity & monotonicity we have...
\[ \mu(U E_i) = \mu(U F_i) \]

\[ = \sum_i \mu(F_i) \quad \text{\textless additivity} \]

\[ \leq \sum_i \mu(E_i) \quad \text{\textless monotonicity} . \]

**Note**

"Uncountable subadditivity" can fail. For example, if \( \mu \) is Lebesgue measure on \( \mathbb{R} \) then

\[ \mu\{x\} = 0 \quad \forall x \in \mathbb{R} \]

so

\[ \mu\left( \bigcup_{x \in \mathbb{R}} \{x\} \right) = \mu(\mathbb{R}) = \infty \]

while

\[ \sum_{x \in \mathbb{R}} \mu\{x\} = \sum_{x \in \mathbb{R}} 0 = 0 \]

Thus

\[ \mu\left( \bigcup_{x \in \mathbb{R}} \{x\} \right) \neq \sum_{x \in \mathbb{R}} \mu\{x\} \]

\[ \uparrow \]

uncountable union!
Basic Properties of Measures: Continuity from Below

If $E_1 \subseteq E_2 \subseteq \ldots$ are nested then we have this picture:

The sets $E_i$ "increase" to $UE_i$. Do their measures increase as well?

Continuity From Below

If $E_1 \in \mathcal{M} \& E_1 \subseteq E_2 \subseteq \ldots$, then

$$
\mu\left( \bigcup_{i=1}^{\infty} E_i \right) = \lim_{i \to \infty} \mu(E_i).
$$

Proof:
Set $E_0 = \emptyset$. Then we have

$$
\bigcup_{i=1}^{\infty} E_i = \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1}) \quad \text{disjoint union!}
$$
Therefore,

\[ \mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \mu \left( \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1}) \right) \]

\[ = \sum_{j=1}^{\infty} \mu (E_j \setminus E_{j-1}) \quad \text{additivity} \]

\[ = \lim_{N \to \infty} \sum_{j=1}^{N} \mu (E_j \setminus E_{j-1}) \]

\[ = \lim_{N \to \infty} \mu \left( \bigcup_{j=1}^{N} (E_j \setminus E_{j-1}) \right) \]

\[ = \lim_{N \to \infty} \mu (E_N). \]

**Note:** We are very tempted to try to create a telescoping sum by writing

\[ \sum \mu (E_j \setminus E_{j-1}) = \sum \mu (E_j) - \sum \mu (E_{j-1}) \]

but we do not know whether \( \mu (E_{j-1}) \) is finite or infinite, so we cannot do this.
Basic Properties of Measures: Continuity From Above

If $E_1 \supseteq E_2 \supseteq \cdots$, then we have this picture:

The sets $E_i$ "decrease" to $\bigcap E_i$. Do their measures decrease as well? NO in general!

**Example**

Let $\mu$ be Lebesgue measure on $\mathbb{R}^d$ — not yet defined, but let us accept on faith that the Lebesgue measure of a ball is its volume.

So, if

$$B_n = \{ x \in \mathbb{R}^d : \|x\| < n \}$$

is the ball of radius $n$ centered at the origin, then $\mu(B_n)$ is its volume, which is finite. Define

$$E_n = B_n^c = \mathbb{R}^d \setminus B_n.$$
Since $\mu(B_n) < \infty$, we have

$$\mu(E_n) = \mu(\mathbb{R}^d) - \mu(B_n) = \infty.$$ 

Thus,

$$\lim_{n \to \infty} \mu(E_n) = \infty.$$ 

Further, since $E_n$ is the "outside" of $B_n$,

$$E_1 \supset E_2 \supset E_3 \supset \cdots$$

and

$$\bigcap_{n=1}^{\infty} E_n = \emptyset.$$ 

Thus,

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\emptyset) = 0 \neq \infty = \lim_{n \to \infty} \mu(E_n).$$

Thus, nested infinite-measure sets can decrease to a finite measure set. On the other hand, if we restrict our attention to finite-measure sets then we do have a nice result.
Continuity From Above

If \( E_i \in M \), \( E_i \supseteq E_2 \supseteq \ldots \), \( \mu(E_i) < \infty \), then

\[
\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_i).
\]

Proof:

Set

\[
F_j = E_1 \setminus E_j
\]

Then

\[
F_1 \subseteq F_2 \subseteq \ldots
\]

Furthermore, since \( \mu(E_1) < \infty \) we have

\[
\mu(F_j) = \mu(E_1) - \mu(E_j).
\]

Also,

\[
\bigcup_{j=1}^{\infty} F_j = E_1 \setminus \left( \bigcap_{i=1}^{\infty} E_i \right)
\]

So

\[
\mu(E_1) - \mu(\bigcap_{i=1}^{\infty} E_i) = \mu(\bigcup_{j=1}^{\infty} F_j)
\]

\[
= \lim_{j \to \infty} \mu(F_j)
\]

\[
= \lim_{j \to \infty} \left( \mu(E_1) - \mu(E_j) \right)
\]

\[
= \mu(E_1) - \lim_{j \to \infty} \mu(E_j) \left( \text{all finite!} \right)
\]

Rearrange.
Note
Instead of assuming \( P(E_1) < \infty \), we just need \( \mu(E_n) < \infty \) for some \( n \in \mathbb{N} \) (why?)
Sets of Measure Zero

As we remarked before, zero-measure sets are very important.

**Terminology: Almost Everywhere.**

A set \( E \subseteq \mathbb{R} \) such that \( \mu(E) = 0 \) is called a null set or a set of measure zero.

Sometimes we may need to be specific about \( \mu \) measure, & in those cases we would speak of a \( \mu \)-null set \( E \).

A property that holds for all \( x \in \mathbb{R} \) except possibly for \( x \) in a null set \( E \) is said to hold almost everywhere (a.e.).

**Example**

Again considering our not-yet-defined Lebesgue measure \( \mu \) on \( \mathbb{R} \), we have \( \mu \{ x \} = 0 \) for each \( x \in \mathbb{R} \). Since \( \mathbb{Q} \) (the rationals) are countable,
we therefore have that

\[ \mu(Q) = \mu \left( \bigcup_{r \in Q} \{ r \} \right) = \sum_{r \in Q} \mu \{ r \} = 0. \]

Thus \( Q \), and indeed any countable set, is a null set w.r.t. Lebesgue measure.

Now consider the characteristic function of \( Q \):

\[ \chi_Q(x) = \begin{cases} 
1, & x \in Q, \\
0, & x \notin Q.
\end{cases} \]

Since \( \chi_Q(x) = 0 \) except for \( x \in Q \), which has zero measure, we write

\[ \chi_Q = 0 \text{ a.e.} \]

That is, \( \chi_Q \) takes the value zero except on a set of measure zero.

Likewise,

\[ \chi_{Q^c} = 1 \text{ a.e.} \]
Note that null sets can be very "large." For example, for the $\delta$-measure we have

$$\delta(\mathbb{R}\setminus\{0\}) = 0.$$ 

Thus, w.r.t. the $\delta$ measure,

$$\chi_{\mathbb{R}\setminus\{0\}} = 0 \text{ a.e.}$$

If we need to explicitly identify $\delta$ measure, we could write "$\delta$-a.e." instead of just "a.e."
Completion of a Measure

Suppose that $E$ is a null set for $\mu$, i.e., $E \in M$ and $\mu(E) = 0$. In general, an arbitrary subset $F \subseteq E$ need not belong to $M$. If it does, then $\mu(F) = 0$ by monotonicity.

If every subset of a null set is measurable, then $\mu$ is said to be complete.

If $\mu$ is not complete, there is a way to extend $\mu$ to a larger $\sigma$-algebra $\overline{M}$ in such a way that the extended $\mu$ is complete.

Theorem: Completion of a Measure

Let $(X, M, \mu)$ be a measure space. Let $\mathcal{N}$ be the collection of null sets:

$$\mathcal{N} = \{ N \in M : \mu(N) = 0 \}.$$

Let

$$\overline{M} = \{ E \cup F : E \in M, F \subseteq N \in \mathcal{N} \}$$

Then:
a. \(\overline{M}\) is a \(\sigma\)-algebra,

b. \exists unique measure \(\bar{\mu}\) on \((X,\overline{M})\) s.t. \(\bar{\mu}|_M = \mu\),

c. \((X,\overline{M},\bar{\mu})\) is complete.

Remark: Thus, we obtain \(\overline{M}\) by unioning all measurable sets \(E \in M\) with all subsets \(F\) of all null sets \(N \in M\).

Essentially, any \(F \subseteq N\) where \(N\) is a null set, "should be" a null set itself. So we are just "throwing in" all \(D\) sets that "should be" null sets into the \(\sigma\)-algebra.

Proof:
a. Since \(M \subseteq \overline{M}\), \(\overline{M}\) is nonempty.

Choose any countable collection of sets \(A_i \in \overline{M}\). Then

\[ A_i = E_i \cup F_i \quad \text{where} \quad E_i \in M \ \& \ F_i \subseteq N_i \in \overline{M}. \]

Since \(\mu(N_i) = 0\), we have \(\mu(U N_i) = 0\)
by subadditivity. Hence

\[ UF_i \subseteq UN_i \in \mathcal{M}. \]

Therefore

\[ U(E_i \cup UF_i) = (UE_i) \cup (UF_i) \subseteq \overline{M}, \]

so \( \overline{M} \) is closed under countable unions.

Exercise: Show \( \overline{M} \) is closed under complements.

6. Given \( E \cup UF \in \overline{M} \), define

\[ \overline{\mu}(E \cup UF) = \mu(E). \]

To show this is well-defined, suppose that

\[ E_1 \cup UF_1 = E_2 \cup UF_2 \text{ where } E_1, E_2 \in M \text{ & } F_i \subseteq N_i \in \mathcal{M}. \]

Then

\[ \mu(E_1) \leq \mu(E_2 \cup UN_2) \quad \text{since } E_1 \subseteq E_2 \cup UF_2 \subseteq E_2 \cup UN_2 \]

\[ \leq \mu(E_2) + \mu(N_2) \quad \text{subadditivity} \]

\[ = \mu(E_2). \]

Similarly \( \mu(E_2) \leq \mu(E_1) \), so \( \mu(E_1) = \mu(E_2) \).
Therefore $\bar{\mu}$ is well-defined.

Exercise: Show $\bar{\mu}$ is a measure, & that $\bar{\mu}$ is the unique measure on $\overline{M}$ that extends $\mu$.

Exercise: Show $\bar{\mu}$ is complete. $\square$