2.5 Product Measures

We want to develop measures & integration on $X \times Y$.

Suppose that $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are measure spaces.

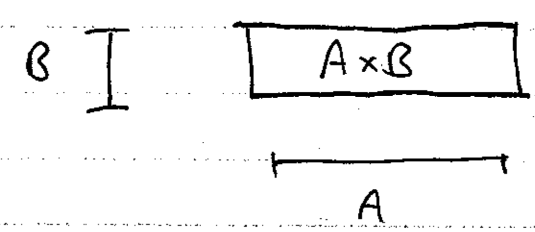
We want to construct an appropriate measure space $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.

What sets will be m & what will their measures be? We'll sketch the construction.

Start with what we certainly want to be m:

If $A \in \mathcal{M}$, $B \in \mathcal{N}$, then we certainly want $A \times B$ to be m. So $\mathcal{M} \otimes \mathcal{N}$ should include all such sets.

Notation
We call $A \times B$ a rectangle, even though this is clearly an abuse of terminology.



We declare that

$$(\mu \times \nu)(A \times B) = \mu(A)\,\nu(B).$$

So now we know how to measure rectangles, and we have to figure out how to extend to arbitrary (m) sets.

If we set

$$\mathcal{E} = \{A \times B : A \in \mathcal{M},\ B \in \mathcal{N}\},$$

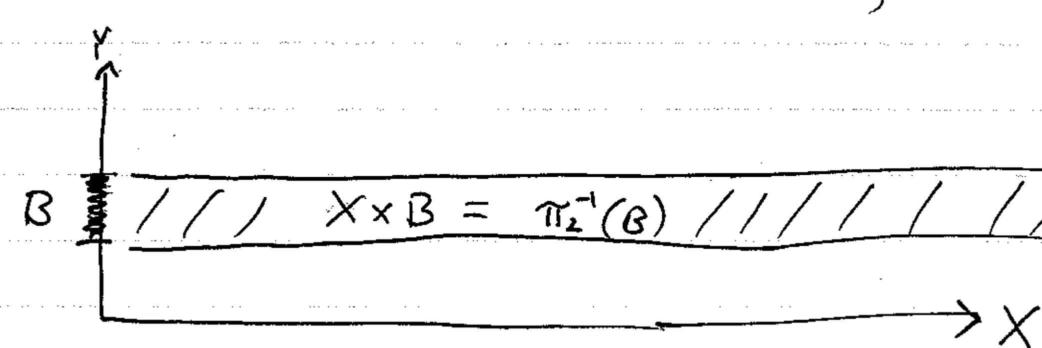
then we define

$$\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{E}),$$ the σ-algebra generated by $\mathcal{E}$.

Exercise

We don't even need all the rectangles. Show that $\mathcal{M} \otimes \mathcal{N}$ is generated by

$$\{A \times Y : A \in \mathcal{M}\} \cup \{X \times B : B \in \mathcal{N}\}$$



Now we construct the measure $\mu \times \nu$ by going through a premeasure construction.

First we form an algebra by taking all finite unions of disjoint algebras:

$$\mathcal{A} = \left\{ \bigcup_{j=1}^{n} A_j \times B_j \; : A_j \in \mathcal{M}, B_j \in \mathcal{N}, \; A_j \times B_j \text{ disjoint} \right\}$$

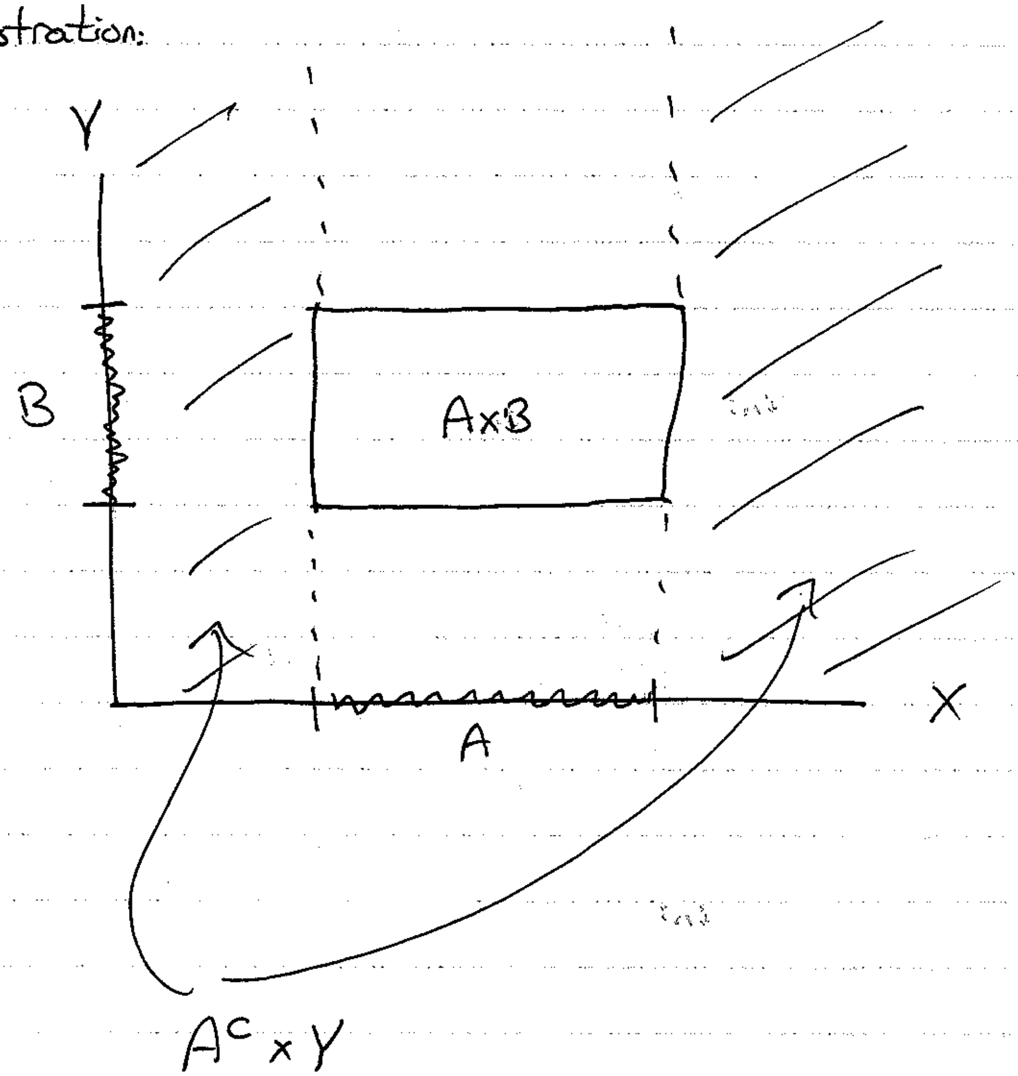
To show this is an algebra, just show that $\mathcal{E}$ is an elementary family, ie.

a. $\emptyset \in \mathcal{E}$

b. $E, F \in \mathcal{E} \implies E \cap F \in \mathcal{E}$

c. $E \in \mathcal{E} \implies E^c = \bigcup_{j=1}^{n} E_j, \; E_j \in \mathcal{M}$ disjoint

Illustration:



$$(A\times B)^c = (A^c \times Y) \cup (A \times B^c) \text{ disjointly.}$$

If $E = \bigcup_{j=1}^{n} A_j \times B_j \in \mathcal{A}$ (disjoint sum of rectangles), then we set

$$\rho(E) = \sum_{j=1}^{n} \mu(A_j)\,\nu(B_j)$$

This defines a premeasure. We then have an outer measure:

$$\rho^*(E) = \inf\left\{ \sum_k \rho(E_k) : E_k \in \mathcal{A},\ E \subseteq \bigcup E_k \right\}$$

Since we've gone through the premeasure process, every set in $\mathcal{M} \otimes \mathcal{N}$ is ρ^*-(m), and

$$\rho^*(A \times B) = \mu(A)\,\nu(B).$$

We then define

$$\mu \times \nu = \rho^*\big|_{\mathcal{M} \otimes \mathcal{N}},$$

this is a measure with σ-algebra $\mathcal{M} \otimes \mathcal{N}$.

Good news: $(\mu \times \nu)(A \times B) = \mu(A)\,\nu(B)$
$\forall A \in \mathcal{M},\ B \in \mathcal{N}.$

In fact, $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ for which $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ holds.

By induction, we can extend to higher-fold products, & show that associativity holds, e.g.,

$$\mathcal{M}_1 \otimes (\mathcal{M}_2 \otimes \mathcal{M}_3) = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$$

$$\mu_1 \times (\mu_2 \times \mu_3) = (\mu_1 \times \mu_2) \times \mu_3$$

We'll take the construction of $\mu \times \nu$ as given from now on.

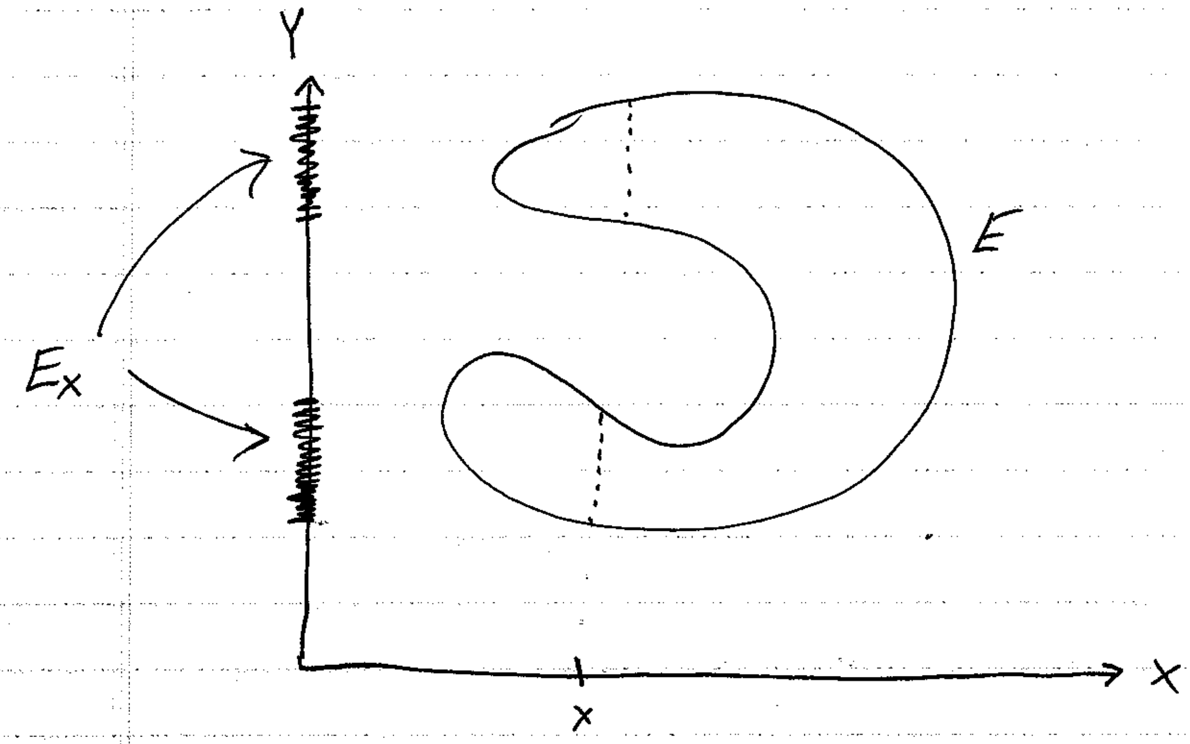
Now we proceed towards theorems involving repeated integration.

Sections of a Set

Given $E \subseteq X \times Y$, $x \in X$, and $y \in Y$,

the x-section of E is $E_x = \{y \in Y : (x,y) \in E\}$

the y-section of E is $E^y = \{x \in X : (x,y) \in E$



Note that $E_x \subseteq Y$ & $E^y \subseteq X$.

Exercises

a. $(E^c)_x = (E_x)^c$, $(E^c)^y = (E^y)^c$

b. $\left(\bigcup E_j\right)_x = \bigcup (E_j)_x$, $\left(\bigcup E_j\right)^y = \bigcup (E_j)^y$

c.
$$(A \times B)_x = \begin{cases} B, & x \in A \\ \emptyset, & x \notin A \end{cases}$$

$$(A \times B)^y = \begin{cases} A, & y \in B \\ \emptyset, & y \notin B \end{cases}$$

Now we show that sections of a (m) set are (m).

Theorem
If $E \in \mathcal{M} \otimes \mathcal{N}$ then $E_x \in \mathcal{N}$ & $E^y \in \mathcal{M}$ $\forall x \in X, y \in Y$.

Proof:
Let
$$\mathcal{R} = \{E \subseteq X \times Y : E_x \in \mathcal{N} \text{ \& } E^y \in \mathcal{M} \ \ \forall x \in X, y \in Y\}$$

Exercise: Use the exercise above to show that $\mathcal{R}$ is a σ-algebra, and that $A \times B \in \mathcal{R}$ whenever $A \in \mathcal{M}$, $B \in \mathcal{N}$. Hence $\mathcal{R}$ contains all rectangles and therefore contains the σ-algebra generated by the rectangles, which is $\mathcal{M} \otimes \mathcal{N}$. That is, $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{R}$, which proves the theorem. $\blacksquare$

Sections of a function

Given $f : X \times Y \to Z$, $x \in X$, $y \in Y$:

the x-section of f is

$$f_x : Y \to Z \qquad \text{i.e., } f_x(y) = f(x,y)$$
$$y \mapsto f(x,y)$$

the y-section of f is

$$f^y : X \longrightarrow Z \qquad \text{i.e., } f^y(x) = f(x,y)$$
$$x \longmapsto f(x,y)$$

Note that

$$(\chi_E)_x = \chi_{E_x} \quad \& \quad (\chi_E)^y = \chi_{E^y}.$$

Exercise

Show that if f is $\mathcal{M}\otimes\mathcal{N}$-(m), then

f_x is $\mathcal{N}$-(m) $\forall x \in X$ and

f^y is $\mathcal{M}$-(m) $\forall y \in Y$.

Motivation: Characteristic Functions

Suppose for the moment that we could switch integrals at will. Then we would have

$$
\begin{aligned}
(\mu\times\nu)(E) &= \iint \chi_E(x,y)\, d(\mu\times\nu)(x,y) \\
&= \int\int \chi_{E_x}(y)\, d\mu(x)\, d\nu(y) \\
&= \int \left(\int \chi_{E_x}(y)\, d\nu(y) \right) d\mu(x) \\
&= \int \nu(E_x)\, d\mu(x).
\end{aligned}
$$

The next theorem will show directly that

$(\mu\times\nu)(E) = \int \nu(E_x)\, d\mu(x)$ is valid.

We begin with finite measures.

<u>Theorem</u>
Suppose $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ are finite measures.

If $E \in \mathcal{M} \otimes \mathcal{N}$, then the following statements hold:

a. $x \longmapsto \nu(E_x)$ is $\mathcal{M}$-m

b. $y \longmapsto \mu(E^y)$ is $\mathcal{N}$-m

c. $(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$

<u>Proof:</u>
Let $\mathcal{C}$ be the set of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which a, b, & c hold.

<u>Claim 1</u>: <u>$\mathcal{C}$ contains all measurable rectangles.</u>

To see this, suppose $E = A \times B$ where $A \in \mathcal{M}$, $B \in \mathcal{N}$.

Then E_x is either B or $\emptyset$, which are m, & likewise E^y is m. Also,

$$\nu(E_x) = \chi_A(x) \cdot \nu(B)$$

so

$$\int \nu(E_x)\, d\mu(x) = \int \chi_A(x)\, \nu(B)\, d\mu(x)$$
$$= \mu(A)\, \nu(B)$$
$$= (\mu \times \nu)(E)$$

and similarly $\int \mu(E^y)\, d\nu(y) = (\mu \times \nu)(E)$.

Hence $E \in \mathcal{C}$.

<u>Exercise:</u> Extend this to show that $\mathcal{A} \subseteq \mathcal{C}$.

Claim 2: $\mathcal{C}$ is closed under increasing unions

Suppose $E_1 \subseteq E_2 \subseteq \cdots$ belong to $\mathcal{C}$, and set $E = \bigcup E_k$. Define

$$f_k(y) = \mu(E_k^y), \quad y \in Y.$$

Since $E_k \in \mathcal{C}$, we have that f_k is m. Since $E_1^y \subseteq E_2^y \subseteq \cdots$ and $\bigcup E_k^y = E^y$, we have by continuity from below that the f_k converge:

$$f(y) = \mu(E^y) = \lim_{k\to\infty} \mu(E_k^y) = \lim_{k\to\infty} f_k(y).$$

Hence f is m. Also,

$$\int \mu(E^y)\, d\nu(y) = \lim_{k\to\infty} \int \mu(E_k^y)\, d\nu(y) \quad \text{MCT}$$

$$= \lim_{k\to\infty} (\mu\times\nu)(E_k) \quad \text{since } E_k \in \mathcal{C}$$

$$= (\mu\times\nu)(E) \quad \text{by continuity from below.}$$

The remaining properties are symmetric, so $E \in \mathcal{C}$.

Claim 3: $\mathcal{C}$ is closed under decreasing intersections

Suppose $E_1 \supseteq E_2 \supseteq \cdots$ belong to $\mathcal{C}$, and set $E = \bigcap E_k$. As before, $f_k(y) = \mu(E_k^y)$ is ⓜ. Also $E_1^y \supseteq E_2^y \supseteq \cdots$ & $E^y = \bigcap E_k^y$. Since our measures are finite, we have by continuity from above that

$$f(y) = \mu(E^y) = \lim_{k\to\infty} \mu(E_k^y) = \lim_{k\to\infty} f_k(y)$$

converges, & therefore is ⓜ. Also, $f_k(y) \searrow f(y) \geq 0$ and all these functions are integrable, since

$$\begin{aligned} \int f_1(y)\, d\nu(y) &= \int \mu(E_1^y)\, d\nu(y) \\ &\leq \int \mu(X)\, d\nu(y) \\ &= \mu(X)\nu(Y) < \infty. \end{aligned}$$

Exercise: $f_k \searrow f \geq 0$ & $f_1 \in L^1(Y) \Rightarrow \int f_k \to \int f$.

Therefore we have

$$\int \mu(E^y)\, d\nu(y) = \lim_{k\to\infty} \int \mu(E_k^y)\, d\nu(y) \quad \text{by the exercise}$$

$$= \lim_{k\to\infty} (\mu\times\nu)(E_k^y) \quad \text{since } E_k \in \mathcal{C}$$

$$= (\mu\times\nu)(E) \quad \text{by continuity from above}$$

Thus $E \in \mathcal{C}$.

Now we appeal to the Monotone Class Lemma: because

i. $\mathcal{A}$ is an algebra & $\mathcal{A} \subseteq \mathcal{C}$,
ii. $\mathcal{C}$ is closed under increasing unions,
iii. $\mathcal{C}$ is closed under decreasing intersections,

it follows that $\mathcal{C}$ contains the σ-algebra generated by $\mathcal{A}$, which is $\mathcal{M}\otimes\mathcal{N}$. ∎

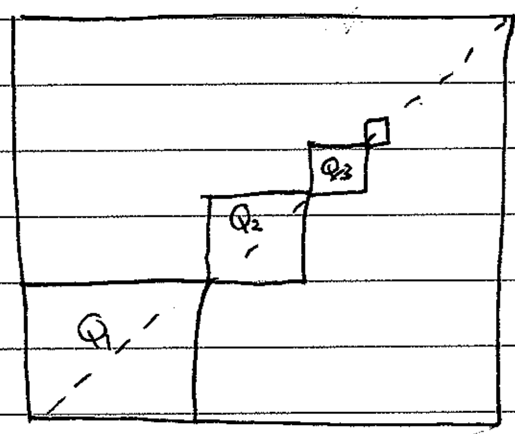
Remark

For a proof of the Monotone Class Lemma, see Folland's text. It actually says that if $\mathcal{S}$ is the smallest monotone class (ie, satisfying i, ii, iii) that contains $\mathcal{A}$, then $\mathcal{S}$ is the σ-algebra generated by $\mathcal{A}$. Hence we conclude that $\mathcal{C} \supseteq \mathcal{S} = \mathcal{M}\otimes\mathcal{N}$.

<u>Exercise:</u> Show that the preceding theorem remains valid if we only assume that μ, ν are σ-finite.

<u>Example</u> (Zygmund)

Let $X = Y = [0,1]$. Place countably many squares on the diagonal of the unit square:



Set $c_n = \frac{1}{|Q_n|}$. Define $f(x,y)$ to be zero outside $\bigcup Q_n$. On each Q_n let f have values as follows:

$$\begin{array}{|c|c|}\hline -c_n & c_n \\ \hline c_n & -c_n \\ \hline \end{array}$$

Q_n

Then f is ⓜ &

$$\int_0^1 f(x,y)\,dy = 0 = \int_0^1 f(x,y)\,dx.$$

Hence the following iterated integrals exist:

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = 0 = \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy.$$

However,

$$\iint_{[0,1]^2} f^+(x,y)\ (dx \times dy) = \sum_{k=1}^{\infty} \frac{1}{2} \frac{1}{|Q_n|} |Q_n| = \infty$$

and likewise $\iint f^- = \infty$. Hence the double integral $\iint f(x,y)\ (dx \times dy)$ does not exist.

Exercise

Show that

$$\int_1^\infty \left(\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\, dy \right) dx = -\frac{\pi}{4}$$

while

$$\int_1^\infty \left(\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\, dx \right) dy = \frac{\pi}{4}$$

Hint: Use the fact that $\frac{d}{dy} \frac{y}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$.

Thus, in general, the iterated integrals

$$\int \left(\int f(x,y)\, d\mu(x) \right) d\nu(y)$$

$$\int \left(\int f(x,y)\, d\nu(y) \right) d\mu(x)$$

and the double integral

$$\iint f(x,y)\, d(\mu\times\nu)(x,y)$$

need not be equal. Tonelli's & Fubini's Theorems will provide hypotheses under which these are all equal.

Tonelli's Theorem says that interchange is allowed for <u>nonnegative</u> functions (if the measure spaces are σ-finite).

<u>Tonelli's Theorem</u>

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.

If $f: X \times Y \to [0, \infty]$ is m, then the following statements hold.

We already know these:

a. $f_x(y) = f(x,y)$ is m for [redacted] $x \in X$.

b. $f^y(x) = f(x,y)$ is m for [redacted] $y \in Y$.

c. $g(x) = \int f_x(y)\, d\nu(y)$ is m

d. $h(y) = \int f^y(x)\, d\mu(x)$ is m

e. As extended real numbers,

$$\iint f(x,y)\, d(\mu\times\nu)(x,y) = \int \left(\int f(x,y)\, d\mu(x) \right) d\nu(y)$$
$$= \int \left(\int f(x,y)\, d\nu(y) \right) d\mu(x).$$

<u>Proof:</u>

Exercise: Verify that our previous results establish Tonelli's Theorem for the case where f is a characteristic function, and extend this by linearity to simple functions.

Given an arbitrary m $f : X \times Y \to [0, \infty]$, let ϕ_n be simple functions $0 \le \phi_n \nearrow f$.

Then $g_n(x) = \int \phi_n(x,y)\, d\nu(y)$ is m, and by the Monotone Convergence Theorem, for each x,

$$g_n(x) = \int \phi_n(x,y)\, d\nu(y) \nearrow \int f(x,y)\, d\nu(y) = g(x),$$

so g is m, & similarly h is m. Also,

$$\iint f \, d(\mu \times \nu) = \lim_{n\to\infty} \iint \phi_n \, d(\mu\times\nu) \quad (\text{MCT})$$

$$= \lim_{n\to\infty} \int \left(\int \phi_n(x,y)\, d\nu(y) \right) d\mu(x)$$

$$= \lim_{n\to\infty} \int g_n \, d\mu$$

$$= \int g \, d\mu \quad (\text{MCT})$$

$$= \int \int f(x,y)\, d\nu(y)\, d\mu(x)$$

and the equality of the other iterated integral is similar. ∎

The following corollary is extremely useful: to test whether $f \in L^1(X \times Y)$, we can check any <u>one</u> of three integrals for finiteness.

<u>Corollary</u>
Let $(X, \mathcal{M}, \mu)$ & $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.

If $f : X \times Y \to \overline{\mathbb{R}}$ or $\mathbb{C}$ is (m), then (as extended real numbers)

$$\iint |f| \, d(\mu \times \nu) = \iint |f(x,y)| \, d\mu(x) \, d\nu(y)$$
$$= \iint |f(x,y)| \, d\nu(y) \, d\mu(x)$$

Consequently, if any one of these integrals is finite, then $f \in L^1(X \times Y)$.

Fubini's Theorem allows the interchange of integrals if f is integrable (thereby again avoiding the ambiguity that is $\infty - \infty$).

<u>Fubini's Theorem</u>

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.

If $f \in L^1(X \times Y)$, then the following statements hold.

a. $f_x \in L^1(Y)$ for μ-a.e. $x \in X$,

b. $f^y \in L^1(X)$ for ν-a.e. $y \in Y$

c. $g(x) = \int f_x(y)\, d\nu(y)$ belongs to $L^1(X)$

d. $h(y) = \int f^y(x)\, d\mu(x)$ belongs to $L^1(Y)$

e. $$\iint f(x,y)\, d(\mu \times \nu)(x,y) = \int \left(\int f(x,y)\, d\mu(x) \right) d\nu(y)$$
$$= \int \left(\int f(x,y)\, d\nu(y) \right) d\mu(x)$$

Proof:

Suppose first that $f \geq 0$. Then by Tonelli's Theorem,

f_x, f^y, g, & h are (m), and

$$0 \leq \int g \, d\mu = \int h \, d\nu = \iint f \, d(\mu \times \nu) < \infty.$$

since $f \in L^1$

Hence $g \in L^1(X)$ & $h \in L^1(Y)$, since they are nonnegative. Consequently, g & h must be finite a.e., so

$$0 \leq g(x) = \int f_x \, d\nu < \infty \quad \text{a.e. } x$$

$$0 \leq h(y) = \int f^y \, d\mu < \infty \quad \text{a.e. } y$$

Therefore f_x is integrable for a.e. x, & f^y is integrable for a.e. y. This completes the proof for the case $f \geq 0$.

For general $f \in L^1(X)$, write

$$f = (f_1 - f_2) + i(f_3 - f_4)$$

where f_1, f_2, f_3, f_4. Then (exercise) apply the preceding case to each f_i, & show that the result follows. ∎

<u>Remark</u>

Technically, elements of L^1 are equivalence classes of functions that are equal a.e. Fubini's Theorem applies to any of the representatives of f.

<u>Remark</u>

Again, remember the Corollary to Tonelli's Theorem, which tells us that to check whether $f \in L^1(X \times Y)$, we can check any <u>one</u> of three integrals, whichever is more convenient.

Unfortunately, $\mu \times \nu$ is rarely complete, so sometimes we need a version of Tonelli/Fubini for the completion of $\mu \times \nu$. For the general case, see Folland's text.

For example, Lebesgue measure on $\mathbb{R}^n$ is the completion of the n-fold product of Lebesgue measure on $\mathbb{R}$. The statement of Tonelli & Fubini for Lebesgue measure is as follows

Tonelli's Theorem

Let $E \subseteq \mathbb{R}^m$ & $F \subseteq \mathbb{R}^n$ be Lebesgue measurable. If $f: E \times F \to [0,\infty]$ is (m), then:

a. f_x is (m) $\forall x \in E$ (Lebesgue measurable)

b. f^y is (m) $\forall y \in F$

c. $g(x) = \int_F f_x(y)\, dy$ is (m) on E

d. $h(y) = \int_E f^y(x)\, dx$ is (m) on F

e. $$\iint_{E \times F} f(x,y)\, (dx\, dy) = \int_F \left(\int_E f(x,y)\, dx \right) dy$$
$$= \int_E \left(\int_F f(x,y)\, dy \right) dx$$

Fubini's Theorem

Let $E \subseteq \mathbb{R}^m$ & $F \subseteq \mathbb{R}^n$ be Lebesgue measurable.

If $f \in L^1(E \times F)$, then:

a. $f_x \in L^1(F)$ for a.e. $x \in E$

b. $f^y \in L^1(E)$ for a.e. $y \in F$

c. $g(x) = \int_F f(x,y)\,dy$ belongs to $L^1(E)$

d. $h(y) = \int_E f(x,y)\,dx$ belongs to $L^1(F)$

e. $$\iint_{E \times F} f(x,y)\,(dx\,dy) = \int_F \left(\int_E f(x,y)\,dx \right) dy$$

$$= \int_E \left(\int_F f(x,y)\,dy \right) dx$$

The Integral as the Area under the Graph

Exercise (See Folland 2.5 #50)
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, & let $f : X \to [0, \infty]$ be ⓜ.

a. The region under the graph of f is

$$G_f = \{(x,y) \in X \times [0,\infty] : 0 \leq y \leq f(x)\}$$

Show that G_f is $\mathcal{M} \times \mathcal{B}$ - measurable
($\mathcal{B}$ = Borel σ-algebra in $\mathbb{R}$)

and that

$$(\mu \times m)(G_f) = \int f \, d\mu$$

(m = Lebesgue measure).

See Folland for a hint.

b. The graph of f is

$$\Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times [0,\infty].$$

Show that Γ_f is ⓜ, & $(\mu \times m)(\Gamma_f) = 0$.

Hint: Consider

$$E_n = \{\varepsilon n \le f < \varepsilon(n+1)\} \quad \& \quad E_\infty = \{f = \infty\}$$

Convolution

Let $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue (m).

a. Show that $f(x)g(y)$ is Lebesgue (m) on $\mathbb{R}^2$.

b. Show $f(x-y)g(y)$ is (m) on $\mathbb{R}^2$.

Hint: Composition of part a with a linear transformation

c. Given $f, g \in L^1(\mathbb{R})$, show their convolution

$$(f*g)(x) = \int f(x-y)\,g(y)\,dy$$

is defined a.e. & is (m).

Hint: Show $\iint |f(x-y)\,g(y)|\,dx\,dy < \infty$, apply Fubini.

d. Show $f*g \in L^1(\mathbb{R})$ & $\|f*g\|_1 \le \|f\|_1 \|g\|_1$

(Thus $L^1(\mathbb{R})$ is a Banach algebra under convolution).