2.6 The n-dimensional Lebesgue integral

We've already developed Lebesgue measure directly on $\mathbb{R}^n$, so we won't do it again. Folland's approach is to define 1-dimensional Lebesgue measure as a special case of Lebesgue-Stieltjes measures, & then create n-dimensional Lebesgue measure as an n-fold product of 1-dimensional Lebesgue measures. The results are the same.

There are a few important issues about Lipschitz transformations that we have not covered yet, so we will do those now.

__Definition__

A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is __Lipschitz__ if

$$\exists C>0 \text{ s.t. } \forall x,y \in \mathbb{R}^n, \qquad \| T(x) - T(y) \| \leq C \| x-y \|.$$

__Remark__

All norms on $\mathbb{R}^n$ are equivalent since $\mathbb{R}^n$ is finite-dimension. Hence if T is Lipschitz w.r.t. one norm, say the Euclidean norm, then it Lipschitz w.r.t. another norm, say the ℓ^1-norm. The constant C will change, but it will still exist. So, we can use whatever norm is most convenient.

<u>Exercise</u>

a. Show that Lipschitz functions are uniformly continuous.

b. Show that if $f: \mathbb{R} \to \mathbb{R}$ is differentiable and f' is bounded, then f is Lipschitz.

Hint: MVT

c. Give an example of a function that is Lipschitz but not differentiable everywhere.

d. Show that $f(x) = x^2$ is not Lipschitz, but it is <u>locally Lipschitz</u>, i.e., Lipschitz on every compact set.

e. Show that every linear function $T: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz.

f. The translation map
$$\tau_a : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto x+a$$
is not linear, but is Lipschitz.

Exercise

Show that Lipschitz mappings preserve Lebesgue measurability.

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz & $E \subseteq \mathbb{R}^n$ is ⓜ, then $T(E)$ is ⓜ.

Hints: T is continuous, and therefore maps compact sets to compact sets. Every ⓜ E can be written $E = F \cup Z$ where F is an F_σ-set and $|Z| = 0$. Show $|T(Z)| = 0$.

Theorem

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear and $E \subseteq \mathbb{R}^n$ is ⓜ, then

$$|T(E)| = |\det(T)| \cdot |E|.$$

Proof:

Let
$$d = |\det(T)|.$$

Given a ⓜ $E \subseteq \mathbb{R}^n$ & $\varepsilon > 0$, $\exists$ cubes Q_k with $E \subseteq \bigcup Q_k$ and

$$|E| \leq \sum_k{}' |Q_k| \leq |E| + \varepsilon.$$

Each $T(Q_k)$ is a parallelepid, and

$$|T(Q_k)| = d\,|Q_k|.$$

Since $T(E) \subseteq \bigcup T(Q_k)$, we have

$$\begin{aligned} |T(E)| &\le {\sum_k}' |T(Q_k)| \\ &= d \, {\sum_k}' |Q_k| \\ &\le d\,|E| + d\varepsilon. \end{aligned}$$

As this is true $\forall \varepsilon > 0$, we have $|T(E)| \le d\,|E|$.

To obtain the converse inequality, let $U \supseteq E$ be an open set with $|U \setminus E| < \varepsilon$. Then we can write $U = \bigcup_k Q_k$ as a countable union of nonoverlapping cubes. Then

$$|U| = {\sum_k}' |Q_k|,$$

and since the $T(Q_k)$ are nonoverlapping parallelepipeds,

$$|T(U)| = {\sum_k}' |T(Q_k)| = d \, {\sum_k}' |Q_k| = d\, m(U).$$

On the other hand,

$$|T(U)| \leq |T(E)| + |T(U \setminus E)|$$

$$\leq |T(E)| + d\,|U \setminus E| \quad \text{by the previous case}$$

$$< |T(E)| + d\varepsilon.$$

Hence

$$|T(E)| > |T(U)| - d\varepsilon$$

$$= d\,|U| - d\varepsilon$$

$$\geq d\,|E| - d\varepsilon.$$

Since ε is arbitrary, we obtain $|T(E)| \geq d\,|E|$. ∎

<u>Remark</u>

We have assumed that the Lebesgue measure of a parallelepiped is equal to its usual volume. This does require proof, but can be shown in a similar manner to how we did it for cubes.

<u>Corollary</u>

Lebesgue measure is invariant under rotations.

<u>Theorem</u>: Linear changes of variable.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear, & let
$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ or $\mathbb{C}$ be (m).

a. $(f \circ T)(x) = f(Tx)$ is (m).

b. If $f \geq 0$ or if $f \in L^1(\mathbb{R}^n)$, then

$$\int f(Tx)\, dx = |\det(T)| \int f(x)\, dx$$

<u>Proof Sketch</u>

Write T as a composition of finitely many:

Coordinate scalings:

$$T(x_1, \dots, x_n) = (x_1, \dots, c x_j, \dots, x_n) \qquad (\det(T) = c)$$

Shears:

$$T(x_1, \dots, x_n) = (x_1, \dots, x_j + c x_k, \dots, x_n) \qquad (\det(T) = +1)$$

Coordinate interchanges:

$$T(x_1, \dots, x_n) = (x_1, \dots, x_k, \dots, x_j, \dots x_n) \qquad (\det(T) = -1).$$

Apply Fubini/Tonelli to each of these, e.g.,

$$\iint f(x_1, x_2 + c x_1)\, dx_1\, dx_2$$

$$= \int \left(\int f(x_1, x_2 + c x_1)\, dx_2 \right) dx_1$$

$$= \int \left(\int f(x_1, x_2)\, dx_2 \right) dx_1$$

etc. Trace through, total effect is multiplication by $|\det(T)|$. $\blacksquare$

<u>Alternative approach</u>

If $f = \chi_E$ then $f \circ T = \chi_{T^{-1}(E)}$. Extend to simple functions & arbitrary functions.