## 3.2 The Lebesgue-Radon-Nikodym Theorem

**Notation**

Recall that if $\mu$ is a positive measure on $(X, \mathcal{M})$ and $f$ is an extended $\mu$-integrable function, then

$$\nu(E) = \int_E f \, d\mu \qquad (*)$$

defines a signed measure on $(X, \mathcal{M})$. Further,

$$\nu^+(E) = \int f^+ \, d\mu, \quad \nu^-(E) = \int f^- \, d\mu,$$

$$|\nu|(E) = \int |f| \, d\mu,$$

and $\nu$ is bounded if $f \in L^1(\nu)$. We will write

$$d\nu = f \, d\mu$$

to mean that $\nu$ is the measure given by $(*)$.
Sometimes it will be convenient to abuse notation a bit and write $\nu = f\, d\mu$ to mean $\nu$ is given by $(*)$.

<u>Definition</u>

A signed measure $\nu$ on $(X, \mathcal{M})$ is <u>absolutely continuous</u> w.r.t. a positive measure $\mu$ on $(X, \mathcal{M})$, denoted $\nu \ll \mu$, if

$$\forall E \in \mathcal{M}, \quad \mu(E) = 0 \implies \nu(E) = 0.$$

<u>Exercise</u>

Show that if $d\nu = f\,d\mu$, then $\nu \ll \mu$.

We will see in the Radon-Nikodym Theorem that <u>all</u> measures absolutely continuous w.r.t. $\mu$ have the form $d\nu = f\,d\mu$ for some $f$.

<u>Exercise</u>

Show that

$$\nu \ll \mu \;\; \& \;\; \nu \perp \mu \implies \nu = 0.$$

The Lebesgue-Radon-Nikodym Theorem will write an arbitrary signed measure $\nu$ as a sum of an absolutely continuous part (w.r.t. $\mu$) & a singular part (w.r.t. $\mu$).

What does "absolute continuity" have to do with continuity?

<u>Theorem</u>

Suppose $\nu$ is a finite signed measure & $\mu$ a positive measure on $(X, \mathcal{M})$. Then TFAE.

a. $\nu \ll \mu$

b. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall E \in \mathcal{M}$, $\mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon$.

Proof:

$\Rightarrow$ Suppose $\nu \geq 0$ & Suppose that statement b failed. Then $\exists \varepsilon > 0$ such that for each $\delta = 2^{-n}$ there exists an $E_n \in \mathcal{M}$ s.t.

$$\mu(E_n) < 2^{-n} \quad \text{but} \quad \nu(E_n) \geq \varepsilon.$$

Let

$$F_k = \bigcup_{n=k}^{\infty} E_n \quad \& \quad F = \bigcap_{k=1}^{\infty} F_k = \limsup_{n \to \infty} E_n.$$

Exercise: $\mu(F) = 0$ & $\nu(F) \geq \varepsilon$

because $\nu \geq 0$ & $\nu$ is finite

Hence $\nu \not\ll \mu$.

*Exercise:* Extend to signed $\nu$.

$\Longleftarrow$ Exercise. $\blacksquare$

Exercise

a. Use the preceding theorem to show that if $\mu \geq 0$ and $f \in L^1(\mu)$, then $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$$\mu(E) < \delta \implies \left| \int_E f \, d\mu \right| < \varepsilon.$$

b. Also give a direct proof of part a by considering

$$f_n(x) = \begin{cases} |f(x)|, & \text{if } |f(x)| \leq n \\ n, & \text{if } |f(x)| > n \end{cases}$$

Note that $f_n \nearrow |f|$.

**Lemma**

Let $\mu, \nu$ be finite positive measures on $(X, \mathcal{M})$. Then either

a. $\nu \perp \mu$, or

b. $\exists \varepsilon > 0$ $\exists E \in \mathcal{M}$ s.t. $\mu(E) > 0$ and

$E$ is a positive set for $\nu - \varepsilon\mu$, ie.,

$$\nu(A) \geq \varepsilon \mu(A) \quad \forall \text{ (m) } A \subseteq E.$$

**Proof:**

For each $n \in \mathbb{N}$, let $X = P_n \cup N_n$ be a Hahn decomposition for the signed measure $\nu - \frac{1}{n}\mu$. Define

$$P = \bigcup P_n, \qquad N = P^c = \bigcap N_n.$$

Then we have that $N$ is a negative set for $\nu - \frac{1}{n}\mu$ for every $n$. Therefore, if $E \subseteq N$ is (m), then

$$0 \leq \nu(E) \leq \frac{1}{n}\mu(E) \quad \forall n \in \mathbb{N},$$

so $\nu(E) = 0$ since $\mu$ is bounded.

Hence $N$ is a null set for $\nu$, which is equivalent to $\nu(N) = 0$ since $\nu \geq 0$.

If $\mu(P) = 0$ then $P$ is a null set for $\mu$, & therefore $\mu \perp \nu$.

On the other hand, if $\mu(P) > 0$ then we must have $\mu(P_n) > 0$ for some $n$. Since $P_n$ is a positive set for $\nu - \frac{1}{n}\mu$, statement b therefore holds with $E = P_n$ & $\varepsilon = \frac{1}{n}$. ∎

<u>Definition</u>
We say that a signed measure $\nu$ is <u>$\sigma$-finite</u> if its total variation $|\nu|$ is $\sigma$-finite.

## Lebesgue-Radon-Nikodym Theorem

Let $\nu$ be a $\sigma$-finite signed measure on $(X, \mathcal{M})$ and $\mu$ a $\sigma$-finite positive measure on $(X, \mathcal{M})$.

a. There exist unique $\sigma$-finite signed measures $\rho, \lambda$ on $(X, \mathcal{M})$ such that

$$\nu = \rho + \lambda, \quad \rho \ll \mu, \quad \lambda \perp \mu.$$

b. There exists an extended $\mu$-integrable function $f$ such that $d\rho = f\,d\mu$, i.e.,

$$\nu = f\,d\mu + \lambda.$$

c. If we also have $\nu = \tilde{f}\,d\mu + \lambda$ where $\tilde{f}$ is an extended $\mu$-integrable function, then

$$\tilde{f} = f \quad \mu\text{-a.e.}$$

Proof:

Case 1: Suppose first that $\mu$ & $\nu$ are both finite, positive measures. Let

$$\mathcal{F} = \left\{ f : X \to [0,\infty] : f \text{ (m)} \ \& \ \int_E f\,d\mu \le \nu(E) \ \forall E \in \mathcal{M} \right\}$$

Since $0 \in \mathcal{F}$ we know $\mathcal{F}$ is nonempty.

Suppose $f, g \in \mathcal{F}$ and set $h = \max\{f, g\}$.

If $E \in \mathcal{M}$, then

$$\int_E h \, d\mu = \int_{E \cap \{f > g\}} f \, d\mu + \int_{E \cap \{g \geq f\}} g \, d\mu$$

$$\leq \nu(E \cap \{f > g\}) + \nu(E \cap \{g \geq f\})$$

$$= \nu(E),$$

so $h \in \mathcal{F}$.

If $f \in \mathcal{F}$, then $0 \leq \int_X f \, d\mu \leq \nu(X) < \infty$, so

$$0 \leq a = \sup\left\{\int f \, d\mu : f \in \mathcal{F}\right\} \leq \nu(X) < \infty.$$

By definition, we can find $f_n \in \mathcal{F}$ such that $\int f_n \, d\mu \longrightarrow a$. Consider

$$g_n = \max\{f_1, \dots, f_n\} \in \mathcal{F}.$$

Define

$$f(x) = \lim_{n \to \infty} g_n(x) = \sup_n f_n(x).$$

Then $f$ is (m), and if $E \in \mathcal{M}$ then since $g_n \uparrow f$ we have

$$\int_E f\,d\mu = \lim_{n\to\infty} \int_E g_n\,d\mu \qquad \text{MCT}$$
$$\leq \nu(E) \qquad \text{since } g_n \in \mathcal{F}$$

and therefore $f \in \mathcal{F}$. Also,

$$a = \lim_{n\to\infty} \int f_n\,d\mu \leq \lim_{n\to\infty} \int g_n\,d\mu = \int f\,d\mu \leq a,$$

so $\int f\,d\mu = a$. Since $f \geq 0$ & $a < \infty$, this implies $0 \leq f < \infty$ $\mu$-a.e. (so by redefining $f$ on a set of measure zero, we can take $f$ to be real-valued everywhere).

We claim now that

$$d\rho = f\,d\mu \quad \text{and} \quad \lambda = \nu - f\,d\mu$$

are the required measures. These certainly are signed measures, and since $f \in \mathcal{F}$ we have

$\rho(E) = \int_E f\,d\mu \le \nu(E) \quad \forall E \in \mathcal{M}$, so $\lambda = \nu - \rho$ is a positive measure. Further, we know $\rho = f\,d\mu \ll \mu$, and $\nu = \rho + \lambda$, so it remains only to show that $\lambda \perp \mu$.

By the lemma, if $\lambda \not\perp \mu$, then $\exists\, \varepsilon > 0$ and $E \in \mathcal{M}$ with $\mu(E) > 0$ such that $E$ is a positive set for $\lambda - \varepsilon\mu$. We claim that $f + \varepsilon\chi_E \in \mathcal{F}$.

To see this, suppose $A \in \mathcal{M}$. Then

$$\int_A f + \varepsilon\chi_E \, d\mu = \int_A f\,d\mu + \varepsilon \int_{A\cap E} d\mu$$

$$= \rho(A) + \varepsilon\mu(A\cap E)$$

$$\le \rho(A) + \lambda(A\cap E)$$

$$\le \rho(A) + \lambda(A) \qquad \text{since } \lambda \ge 0$$

$$= \nu(A).$$

Thus we have $f + \varepsilon\chi_E \in \mathcal{F}$. But then

$$
\begin{aligned}
a &= \int f\,d\mu \\
&< \int f\,d\mu + \varepsilon\mu(E) && \text{since } \mu(E) > 0 \\
&= \int f + \varepsilon\chi_E\,d\mu \\
&\leq a && \text{since } f + \varepsilon\chi_E \in \mathcal{F},
\end{aligned}
$$

which is a contradiction. Hence we must indeed have $\lambda \perp \mu$.

Now we show uniqueness. Suppose that we also had $\nu = \rho' + \lambda'$ with $\rho' \ll \mu$ & $\lambda' \perp \mu$. Then since $\nu = \rho + \lambda$, we have $\lambda - \lambda' = \rho' - \rho$. But

$$\lambda - \lambda' \perp \mu \quad \& \quad \rho' - \rho \ll \mu,$$

which implies $\lambda - \lambda' = 0 = \rho' - \rho$.

Also, if $d\rho = \tilde{f}\,d\mu = f\,d\mu$, then

$$\int_E (f - \tilde{f})\,d\mu = 0 \quad \forall E \in \mathcal{M},$$

which implies by an earlier result that $f = \tilde{f}$ $\mu$-a.e.

Case 2: Suppose that $\mu, \nu$ are both $\sigma$-finite positive measures.

By applying the disjointization trick, we can write $X = \bigcup E_j$ and $X = \bigcup F_k$ as disjoint unions with $\mu(E_j) < \infty$, $\nu(F_k) < \infty$. Then

$$X = \bigcup_{j,k} (E_j \cap F_k) = \bigcup_{\ell} A_\ell \text{ disjointly}$$

with $\mu(A_\ell), \nu(A_\ell) < \infty$ $\forall \ell$. Define

$$\mu_k(E) = \mu(E \cap A_k), \quad \nu_k(E) = \nu(E \cap A_k).$$

Then each $\mu_k, \nu_k$ is a finite positive measure, so by Case 1 we can write $\nu_k = \rho_k + \lambda_k$ for some unique measures with $\rho_k \ll \mu_k$ & $\lambda_k \perp \mu_k$.

Note that

$$\mu_k(A_k^c) = \mu(A_k^c \cap A_k) = 0,$$

so $A_k^c$ is a $\mu_k$-null set. Therefore

$$f_k' = f_k \cdot \chi_{A_k}$$

equals $f_k$ $\mu_k$-a.e., so we can replace $f_k$ with $f_k'$ without changing $\lambda_k$ or $\rho_k$. In other words, we can assume that $f_k(x) = 0$ $\forall x \notin A_k$.

Since the $A_k$ are disjoint, we can therefore define

$$f = \sum_{k=1}^{\infty} f_k .$$

Since $f \geq 0$,

$$d\rho = f\, d\mu$$

defines a positive measure. Also,

$$\lambda = \sum_{k=1}^{\infty} \lambda_k$$

is a positive measure, since each $\lambda_k \geq 0$.

Exercises: $\lambda, \rho$ are $\sigma$-finite, $\nu = \rho + \lambda$, $\rho \ll \mu$, $\lambda \perp \mu$, & the uniqueness statements hold.

Case 3: Exercise: Extend to an arbitrary signed $\sigma$-finite measure $\nu$ by considering $\nu^+$ & $\nu^-$. ∎

## Notation

We refer to $\nu = \rho + \lambda$ as the Lebesgue decomposition of $\nu$ w.r.t. $\mu$.

The special case where $\nu \ll \mu$ (i.e., $\lambda = 0$) is important.

## Radon-Nikodym Theorem

If $\nu$ is a $\sigma$-finite signed measure on $(X, \mathcal{M})$ and $\mu$ is a $\sigma$-finite positive measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$, then $\exists$ extended $\mu$-integrable function $f$ such that $d\nu = f\,d\mu$. Any two functions with this property are equal $\mu$-a.e.

*Notation*

If $\nu \ll \mu$, then the function $f$ s.t. $d\nu = f\,d\mu$ is called the **Radon-Nikodym derivative** of $\nu$ w.r.t. $\mu$, and is often denoted

$$f = \frac{d\nu}{d\mu},$$

i.e.,

$$d\nu = \frac{d\nu}{d\mu}\,d\mu.$$

It is unique up to sets of $\mu$-measure zero.

*Note*

By an earlier exercise, if $d\nu = f\,d\mu$, then

$d|\nu| = |f|\,d\mu$, i.e., $d|\nu| = \left|\frac{d\nu}{d\mu}\right| d\mu$.

*Exercise*

Let $\nu$ be a $\sigma$-finite signed measure & $\mu, \lambda$ $\sigma$-finite positive measures. Show that

$$\nu \ll \mu \ \& \ \mu \ll \lambda \Rightarrow \nu \ll \lambda$$

and, in this case, if $d\nu = f\,d\mu$ & $d\mu = g\,d\lambda$, then $d\nu = fg\,d\lambda$. In other words,

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}.$$

Remark

We know that $\delta \not\ll dx$, ie., $\delta$ is not absolutely continuous w.r.t. Lebesgue measure. Pretend that we did have $\delta \ll dx$. Then there would exist a Radon-Nikodym derivative, a function that we will call $\delta(x)$ that satisfies $d\delta = \delta(x)\,dx$. Then

$$f(0) \underset{\substack{\uparrow \\ \text{earlier exercise}}}{=} \int f\,d\delta = \int f(x)\delta(x)\,dx$$

There is no such function $\delta(x)$, but it is common to abuse notation & write $\int f(x)\,\delta(x)\,dx = f(0)$. However, what is really meant is not an integral w.r.t. Lebesgue measure $dx$ but rather the integral

$\int f \, d\delta = \int f(x) \, d\delta(x)$ w.r.t. the $\delta$-measure.

Exercise

Let $\mu$ denote counting measure on $\mathbb{R}$, which is not $\sigma$-finite.

a. Show that $dx \ll \mu$, but $dx \neq f \, d\mu$ for any function $f$.

b. Prove that $\mu$ has no Lebesgue decomposition w.r.t. $dx$, i.e., $\nexists$ signed measures $\rho, \lambda$ with $\mu = \rho + \lambda$, $\rho \ll dx$, & $\lambda \perp dx$.