3.3 Complex Measures

Definition

A function $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ is a complex measure on a measurable space $(X, \mathcal{M})$ if

a. $\nu(\emptyset) = 0$

b. If $E_1, E_2, \ldots \in \mathcal{M}$ are disjoint, then

$$\nu\left(\cup E_k\right) = \sum_k{}' \nu(E_k).$$

Remark

As was the case for signed measures, since the union $\cup E_k$ is independent of order, the series $\sum' \nu(E_k)$ must converge unconditionally, & therefore must converge absolutely, i.e., $\sum' |\nu(E_k)|$.

Note

A complex measure can only take complex values, & hence all complex measures are finite measures by definition.

Furthermore, they are bounded measures in the following sense.

Exercise/Notation

Let ν be a complex measure. Define

$$\nu_r(E) = \operatorname{Re} \nu(E) \quad \& \quad \nu_i(E) = \operatorname{Im} \nu(E), \qquad E \in \mathcal{M}.$$

Show that ν_r, ν_i are bounded signed measures, & for $E \in \mathcal{M}$ we have

$$|\nu(E)| \leq |\nu_r(E)| + |\nu_i(E)| \leq |\nu_r|(X) + |\nu_i|(X)$$

Therefore ν is bounded in the sense that

$$\sup_{E \in \mathcal{M}} |\nu(E)| < \infty.$$

We refer to ν_r as the <u>real part</u> of ν, and ν_i as the <u>imaginary part</u>.

Definition: Integration

If ν is a complex measure, then we set

$$L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i) = L^1(|\nu_r|) \cap L^1(|\nu_i|).$$

If $f \in L^1(\nu)$, then

$$\int f \, d\nu = \int f \, d\nu_r + i \int f \, d\nu_i$$

Note that $\int f \, d\nu$ is a well-defined complex scalar.

Most definitions likewise carry over by appling them to the real & imaginary parts.

Definition

A complex measure ν on $(X, \mathcal{M})$ is absolutely continuous w.r.t. a positive measure μ on $(X, \mathcal{M})$, denoted $\nu \ll \mu$, if

$$\forall E \in \mathcal{M}, \quad \mu(E) = 0 \Rightarrow \nu(E) = 0$$

Equivalently, $\nu \ll \mu$ if $\nu_r \ll \mu$ & $\nu_i \ll \mu$.

Exercise

If μ is a positive measure & $g \in L^1(\mu)$ (complex version), then $d\nu = g\, d\mu$ defines a complex measure & $\nu \ll \mu$.

Definition

A complex measure is *singular* w.r.t. another complex measure μ, denoted $\nu \perp \mu$, if

$$\nu_r \perp \mu_r, \quad \nu_r \perp \mu_i, \quad \nu_i \perp \mu_r, \quad \nu_i \perp \mu_i.$$

Lebesgue-Radon-Nikodym Theorem

Let ν be a complex measure on $(X, \mathcal{M})$ and μ a σ-finite positive measure on $(X, \mathcal{M})$.
Then $\exists f \in L^1(\mu)$ & a complex measure λ s.t.

$$\nu = f\, d\mu + \lambda, \qquad \lambda \perp \mu.$$

If we also have $\nu = \tilde{f}\, d\mu + \tilde{\lambda}$ where $\tilde{f} \in L^1(\mu)$ and $\tilde{\lambda} \perp \mu$, then

$$\tilde{\lambda} = \lambda \quad \& \quad \tilde{f} = f \ \mu\text{-a.e.}$$

We need the following exercise in order to define the total variation of a complex measure.

Exercise

Let ν be a complex measure on $(X, \mathcal{M})$. Let

$$\mu = |\nu_r| + |\nu_i|,$$

and show that μ is a bounded positive measure with $\nu \ll \mu$. Conclude that $d\nu = f\,d\mu$ for some $f \in L^1(\mu)$.

Remark

The total variation of a complex measure is a little more awkward to define than it is for a signed measure. By the preceding exercise, there is at least one positive measure μ & a function $f \in L^1(\mu)$ such that $d\nu = f\,d\mu$, and we define the total variation $|\nu|$ of ν to be the measure

$$d|\nu| = |f|\,d\mu.$$

However, we want this definition to be independent of the choice of μ & f, which is the content of the following theorem.

<u>Theorem</u>

Let ν be a complex measure on $(X, \mathcal{M})$.

If μ_1, μ_2 are bounded positive measures & $f_1 \in L^1(\mu_1)$, $f_2 \in L^2(\mu_2)$ are such that

$$f_1 \, d\mu_1 = d\nu = f_2 \, d\mu_2,$$

then $|f_1| \, d\mu_1 = |f_2| \, d\mu_2$.

<u>Proof:</u>

Since $\mu_1 \ll \mu_1 + \mu_2$, $\exists \, g_1 \in L^1(\mu_1 + \mu_2)$ s.t. $d\mu_1 = g_1 \, d(\mu_1 + \mu_2)$. Likewise $\exists \, g_2 \in L^1(\mu_1 + \mu_2)$ s.t. $d\mu_2 = g_2 \, d(\mu_1 + \mu_2)$. Because $\mu_1, \mu_2,$ & $\mu_1 + \mu_2$ are all positive measures, we have $g_1, g_2 \geq 0$ $(\mu_1 + \mu_2)$-a.e.

Now, by a previous exercise, since

$d\nu = f_1 \, d\mu_1$ & $d\mu_1 = g_1 \, d(\mu_1 + \mu_2)$, we have

$$d\nu = f_1 g_1 \, d(\mu_1 + \mu_2).$$

Likewise

$$d\nu = f_2 g_2 \, d(\mu_1 + \mu_2).$$

But then the uniqueness clause of the Lebesgue–Radon–Nikodym Theorem implies that

$$f_1 g_1 = f_2 g_2 \quad (\mu_1+\mu_2)\text{-a.e.}$$

Consequently,

$$|f_1|\, g_1 = |f_1 g_1| = |f_2 g_2| = |f_2|\, g_2 \quad (\mu_1+\mu_2)\text{-a.e.},$$

and hence

$$|f_1|\, d\mu_1 = |f_1|\, g_1 \, d(\mu_1+\mu_2)$$

$$= |f_2|\, g_2 \, d(\mu_1+\mu_2)$$

$$= |f_2|\, d\mu_2 . \quad \blacksquare$$

<u>Definition</u>

Let ν be a complex measure on $(X, \mathcal{M})$. Then the <u>total variation</u> $|\nu|$ of ν is the positive measure $d|\nu| = |f|\, d\mu$ where μ is any positive bounded measure and f any function in $L^1(\mu)$ such that $d\nu = f\, d\mu$.

<u>Exercise</u>

Let ν be a complex measure on $(X, \mathcal{M})$.

a. Show $|\nu(E)| \leq |\nu|(E) \quad \forall E \in \mathcal{M}$.

b. Show $\nu \ll |\nu|$, & $\exists\, g$ with $|g| = 1$ $|\nu|$-a.e. such that $d\nu = g\, d|\nu|$. (This is called the <u>polar decomposition</u> of ν.)

c. If $f \in L^1(\nu)$, then $\left|\int f\, d\nu\right| \leq \int |f|\, d|\nu|$.

The following exercise gives some equivalent reformulations of the total variation that are sometimes easier to apply than our definition. In many texts, one of these forms is taken as the definition of $|\nu|$, and the other forms are derived from it.

<u>Hints</u>

a. By definition of total variation, $d|\nu| = |f|\, d\mu$ where $\mu \geq 0$, $f \in L^1(\mu)$, & $d\nu = f\, d\mu$. Hence $|\nu(E)| = \left|\int_E f\, d\mu\right|$.

b. Part a implies $\nu \ll |\nu|$. So $d\nu = g\, d|\nu|$ for some $g \in L^1(|\nu|)$. Then $d|\nu| = |g|\, d|\nu|$. Show $\int_E (|g| - 1)\, d|\nu| = 0 \quad \forall E \in \mathcal{M}$. Use this to show $|g| = 1$ $|\nu|$-a.e.

Exercise

If ν is a complex measure on $(X, \mathcal{M})$, then:

a. $$|\nu|(E) = \sup\left\{ \sum_{k=1}^{n}{}' |\nu(E_k)| : n \in \mathbb{N},\ E_k \in \mathcal{M},\ E = \bigcup_{k=1}^{n} E_k \text{ disjointly} \right\}$$

b. $$|\nu|(E) = \sup\left\{ \sum_{k=1}^{\infty}{}' |\nu(E_k)| : E_k \in \mathcal{M},\ E = \bigcup_{k=1}^{\infty} E_k \text{ disjointly} \right\}$$

c. $$|\nu|(E) = \sup\left\{ \left| \int_E f \, d\nu \right| : |f| \le 1 \ \ |\nu|\text{-a.e.} \right\}$$

Hints:

Let $\mu_1(E)$, $\mu_2(E)$, $\mu_3(E)$ denote the quantities defined on the right-hand side of a, b, c.
Show that $\mu_1 \le \mu_2 \le \mu_3 = |\nu|$ & $\mu_3 \le \mu_1$.

More hints.

$\mu_2 \le \mu_3$: Suppose $E = \bigcup_{k=1}^{\infty} E_k$ disjointly. Let c_k be the scalars of unit modulus such that
$\left| \int_{E_k} d\nu \right| = c_k \int_{E_k} d\nu$, and set $f = \sum' c_k \chi_{E_k}$.

$|\nu| \le \mu_3$: $\exists f$ with $|f| = 1$ ν-a.e. such that $d\nu = f \, d|\nu|$.
Then $\bar{f} \, d\nu = \bar{f} f \, d|\nu| = |f|^2 d|\nu| = d|\nu|$.

$\mu_3 \le \mu_1$: Suppose $\|f\| \le 1$. Since f is bounded, $\exists$ simple functions that converge to f uniformly. Hence if $\varepsilon > 0$, then $\exists$ $\phi = \sum_{j=1}^{N} c_j \chi_{E_j}$ such that

$$\sup_{x \in X} |f(x) - \phi(x)| \le \varepsilon. \quad \text{Consequently}$$

$|c_j| \le 1 + \varepsilon$ since $c_j = \phi(x)$ for some x,

and $\left| \int f \, d\nu \right| \le \left| \int \phi \, d\nu \right| + \varepsilon \, \|\nu\|$.

Exercise

Let ν be a complex measure. Show

$$E \in \mathcal{M} \text{ is a } \nu\text{-null set} \iff |\nu|(E) = 0.$$

Notation

For a complex measure ν, we say a property holds ν-a.e. if it holds except on a null set for ν. Hence ν-a.e. is the same as $|\nu|$-a.e.

Exercise

Let

$$M_b(X) = \{\nu : \nu \text{ is a complex measure on } (X, \mathcal{M})\}.$$

a. Show that $M_b(X)$ is a (complex) vector space, and that

$$\|\nu\| = |\nu|(X)$$

is a norm on $M_b(X)$.

b. Show that $M_b(X)$ is a Banach space, i.e., it is complete w.r.t. the norm $\|\cdot\|$ (this is not trivial, don't worry if you don't get it completely).

Hint: Suppose $\{\nu_n\}_{n \in \mathbb{N}}$ is Cauchy in $M_b(X)$.

Fix $E \in \mathcal{M}$. Show $\{\nu_n(E)\}_{n \in \mathbb{N}}$ is a Cauchy sequence of complex scalars. Define $\nu(E) = \lim_{n\to\infty} \nu_n(E)$.

Show ν is a complex measure. Hard part:

Show $\|\nu - \nu_n\| \to 0$.

<u>Exercise</u>

Let μ be a positive measure on $(X, \mathcal{M})$. Show that if $f \in L^1(\mu)$ & $d\nu = f\,d\mu$, then

$$\|\nu\| = \|f\|_1 = \int |f|\,d\mu.$$

Consequently, $f \longmapsto f\,d\mu$ is an isometric embedding of $L^1(\mu)$ into $M_b(X)$.

Exercises

1. Show that if $f: X \to \mathbb{C}$ is (m), then $\exists$ simple functions ϕ_k such that $\phi_k(x) \to f(x)$ pointwise, and $|\phi_k(x)| \leq |f(x)|$ $\forall x \in X$, $k \in \mathbb{N}$.

2. Let μ be a positive measure on $(X, \mathcal{M})$, & fix $g \in L^1(\mu)$. Define $d\nu = g\, d\mu$. Show that if $f \in L^1(\nu)$, then $\int f\, d\nu = \int fg\, d\mu$.

3. Show that if ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.
   Hint: Consider the polar form of ν.

4. Let ν be a complex measure & μ a positive measure on $(X, \mathcal{M})$. Show that

$$\nu \ll \mu \iff |\nu| \ll \mu.$$

5. Define the complex conjugate of a complex measure ν by

$$\bar{\nu}(E) = \overline{\nu(E)}, \qquad E \in \mathcal{M}.$$

Show that $\bar{\nu}$ is a complex measure, and

$$\int f\, d\bar{\nu} = \overline{\int \bar{f}\, d\nu} \quad \text{for } f \in L^1(\nu).$$

6. Given a complex measure ν on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, find an explicit description of $|\nu|$.

7. Show that

$$\nu \longmapsto (\nu\{k\})_{k \in \mathbb{N}}$$

is an isometric isomorphism of $M_b(\mathbb{N})$ onto $\ell^1(\mathbb{N})$. Thus $M_b(\mathbb{N}) \cong \ell^1(\mathbb{N})$.