Hints for Exercises and Additional Problems

Exercises from Chapter 1

1.12 Proceed by induction. Equation (1.2) holds trivially if $N = 1$. Suppose that equation (1.2) holds for some integer $N \geq 1$. Let Q and $Q_1, \dots, Q_N, Q_{N+1}$ be boxes such that $Q \subseteq Q_1 \cup \cdots \cup Q_N \cup Q_{N+1}$. Allowing degenerate boxes, $Q \cap Q_k$ is a box and $\operatorname{vol}(Q \cap Q_k) \leq \operatorname{vol}(Q_k)$. Therefore we can replace each Q_k by $Q \cap Q_k$, i.e., we can assume that $Q_k \subseteq Q$ for each k. Further, since a degenerate box has volume zero, we can assume that Q_k is nondegenerate for each k. Write $Q \backslash Q_{N+1}$ as a union of finitely many boxes $K_1, \dots, K_m$ that intersect only along their boundaries, and use the inductive hypothesis.

1.44 Write $E_1 = \left(\bigcap_{k=1}^{\infty} E_k\right) \cup (E_1 \backslash E_2) \cup (E_2 \backslash E_3) \cup \cdots$, and then apply countable additivity in a similar fashion to how it was used in the proof of Theorem 1.44.

1.53 (a) Here is an approach that only uses crude estimates.

Step 1. Let Q be a *cube* in $\mathbb{R}^d$ with side lengths s. The diameter of Q is $n^{1/2}s$, so, since T is Lipschitz, the diameter of $T(Q)$ is at most $Cn^{1/2}s$. Conclude that $|T(Q)|_e \leq C^n n^{n/2}\,|Q|$.

Step 2. Now let Q be a *box* in $\mathbb{R}^d$. Given $\varepsilon > 0$, show that there exists a box $\widetilde{Q} \supseteq Q$ that is a union of finitely many nonoverlapping cubes and satisfies $|\widetilde{Q}| \leq |Q| + \varepsilon$.

Step 3. If $|Z| = 0$ then there exist boxes Q_k that cover Z and satisfy $\sum |Q_k| < \varepsilon$. Use Step 2 to find a box $\widetilde{Q}_k$ that is a finite union of cubes and satisfies $|\widetilde{Q}_k| \leq |Q_k| + 2^{-k}\varepsilon$.

Additional Problems from Chapter 1

1.5 (a) This is a special case of part (b), but it may be easier to work this part first. Cover with squares having small sidelengths.

(b) The exterior measure is zero.

1.7 (d) Consider $U = (0,1)\backslash C$.

1.9 Suppose that $S = \{x_1, x_2, \dots\}$ is a perfect subset of $\mathbb{R}^d$. Let $n_1 = 1$ and $r_1 = 1$, and let $U_1 = B_{r_1}(x_{n_1})$ be the the open ball of radius r_1 centered at x_{n_1}. Let n_2 be the first integer greater than n_1 such that $x_{n_2} \in U_1$, and let $U_2 = B_{r_2}(x_{n_2})$ be such that $U_2 \subseteq \overline{U_2} \subseteq U_1$ but $x_{n_1} \notin U_2$. Continue in this way, and then define $K = \cap(\overline{U_n} \cap S)$. The sets $\overline{U_n} \cap S$ are compact and nested decreasing, so the Cantor Intersection Theorem implies that K is nonempty. Show that no element of S can belong to K.

1.14 (b) $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a G_δ set.

1.15 Consider $E = \big(\mathbb{Q} \cap [0,1]\big) \cup \big((\mathbb{R}\backslash\mathbb{Q}) \cap [2,3]\big)$.

1.16 Remark: This problem generalizes without change from the domain $\mathbb{R}^d$ to the setting of a generic metric space X.

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1.18 (b) Define $f(x) = 0$ if x is irrational. If x is a nonzero rational, write $x = p/q$ in lowest terms and define $f(x) = 1/q$. Finally, set $f(0) = 1$. Show that f is continuous at each irrational point, and discontinuous at each rational.

1.20 (a) $\Rightarrow$ (b). Use the fact that every measurable set is "almost" an open set, and every open set is "almost" a union of finitely many nonoverlapping boxes (in fact, every open set is a union of countably many nonoverlapping boxes).

(b) $\Leftarrow$ (a). Let $U \supseteq S$ and $V \supseteq N_1$ be open sets such that $|U\backslash S| \le \varepsilon$ and $|V| \le |N_1|_e + \varepsilon$ (note that N_1, N_2 need not be measurable). Then $G = U \cup V$ is open and $G \supseteq E$.

1.23 Use upper and lower Riemann sums to approximate the given regions, then apply finite additivity and monotonicity of Lebesgue measure.

1.25 Given $a_k \in \mathbb{C}$, we say that $\prod_{k=1}^\infty a_k = L$ if $\lim_{N\to\infty} \prod_{k=1}^N a_k = L$ (if $L = 0$ then the infinite product is often said to *diverge to zero*). If $0 < \theta_k < 1$ then the following statements hold (these are usually proved in a course on complex analysis).

(a) If $\sum_{k=1}^\infty \theta_k = \infty$ then $\prod_{k=1}^\infty (1-\theta_k) = 0$.

(b) If $\sum_{k=1}^\infty \theta_k < \infty$ then $\prod_{k=1}^\infty (1-\theta_k)$ converges to a positive value.

1.30 Let N be a nonmeasurable subset of $[0,1]$ whose rational translates are disjoint, and consider $\{N + r\}_{r\in\mathbb{Q}\cap[0,1]}$.