Hints for Exercises and Additional Problems

Additional Problems from Chapter 2

2.5 Let $\mathcal{U}$ denote the set of all open subsets of $\mathbb{R}$. Any $U \in \mathcal{U}$ can be written as $U = \cup(a_k, b_k) \in \Sigma(\mathcal{E}_1)$, so $\mathcal{U} \subseteq \Sigma(\mathcal{E}_1)$. But $\Sigma(\mathcal{U})$ is the smallest σ-algebra that contains $\mathcal{U}$, so $\mathcal{B}_{\mathbb{R}} = \Sigma(\mathcal{U}) \subseteq \Sigma(\mathcal{E}_1)$.

(b) Given $a < b$,

$$[a, b] \;=\; \bigcap_{k=1}^{\infty} \left(a - \tfrac{1}{k}, b + \tfrac{1}{k}\right) \;\in\; \Sigma(\mathcal{E}_1).$$

Hence $\mathcal{E}_2 \subseteq \Sigma(\mathcal{E}_1)$, so $\Sigma(\mathcal{E}_2) \subseteq \Sigma(\mathcal{E}_1)$.

(f) Given $a \in \mathbb{R}$,

$$[a, \infty) \;=\; \bigcap_{r\in\mathbb{Q}, r<a} (r, \infty) \;\in\; \Sigma(\mathcal{E}_6).$$

2.7 If Σ contains infinitely many *disjoint* sets $E_1, E_2, \dots$ then Σ must be uncountable. One method of showing the existence of such sets is to define a relation on X by declaring $x \sim y$ if and only if

$$\forall\, A \in \Sigma, \quad x \in A \Leftrightarrow y \in A.$$

Prove that $\sim$ is an equivalence relation, and show that if Σ is countable then the equivalence classes $[x] \;=\; \cap\{A \in \Sigma : x \in A\}$ all belong to Σ.

2.9 Let $X = \{x_n\}_{n\in\mathbb{N}}$. Let Σ_N consist of every set $A \in \mathcal{P}(\{x_1, \dots, x_N\})$ together with the complement of each such set A.

2.17 Write $E = \cup E_n$ where $E_n \;=\; \{x \in X : \mu\{x\} > \frac{1}{n}\}$. How many points can be in E_n?

2.19 Define

$$C \;=\; \sup\{\mu(F) : F \in \Sigma, F \subseteq E, \mu(F) < \infty\}.$$

If $C < \infty$ then there exist measurable sets $F_k \subseteq E$ with finite measure such that $\mu(F_k) \to C$. Consider the sets $F = \cup F_k$ and $A = E\backslash F$.