Hints for Exercises and Additional Problems

Exercises from Chapter 3

3.3 Let $\mathcal{F} = \{F \subseteq Y : f^{-1}(F) \in \Sigma_X\}$, and show that $\mathcal{F}$ is a σ-algebra on Y.

3.15 (a) $\Leftrightarrow$ (b). Show that $\Sigma = \bigl\{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\bigr\}$ is a σ-algebra on $\overline{\mathbb{R}}$.

3.43 Prove the results for nonnegative functions first, and then write $f = f^+ - f^-$, $g = g^+ - g^-$.

Additional Problems from Chapter 3

3.2 Given an arbitrary $a \in \mathbb{R}$, find $a_k \in A$ such that $\{f > a\} = \cup\{f > a_k\}$.

3.3 For a counterexample, let N be a nonmeasurable subset of $\mathbb{R}$ and define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ if $x \in N$, and $f(x) = -x$ if $x \notin N$.

3.5 Consider $E_k = \{f > k\}$.

3.7 Consider $f(x) = \sup_k |f_k(x)|$ and $E_k = \{f \le k\}$.

3.10 Consider Problems 1.36 and 2.10.

3.12 Consider $F_1(x,y) = f(x)$.

3.15 (b) Suppose that f is continuous at almost every point. Since the functions $g_n = f\,\chi_{[-n,n]}$ converge pointwise to f, it suffices to show that each function g_n is measurable. In other words, we can assume that f is zero outside of some finite interval $[-n,n]$. Show that there exist step functions

$$f_k \;=\; \sum_{j=1}^{N_k} f(x_j^k)\,\chi_{[x_{j-1}^k,\,x_j^k)}$$

that converge to f at almost every point.

(c) The function $\chi_{[0,1]}$ is continuous a.e. but does not equal any continuous function a.e.

3.16 If $f_k(x) = k\bigl(f(x+\frac{1}{k}) - f(x)\bigr)$, then $f_k(x)$ converges to $f'(x)$ at each point where f is differentiable.

3.20 (c) Proceed through cases. Case 1: $f = g = 0$ a.e. Case 2: $g_k = g$ for every k. Case 3: Arbtrary f_k, g_k.

3.22 To prove the Triangle Inequality, first show that if a, b, $c \ge 0$ and $a \le b + c$, then

$$\frac{a}{1+a} \;\le\; \frac{b}{1+b} + \frac{c}{1+c}.$$