MATH 6337 (C. Heil) HOMEWORK #6 Due date: April 12, 2011

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

Throughout this homework, we assume that (X, Σ, μ) is a measure space.

1. Let $f_n, f \colon X \to [0,\infty]$ be measurable functions such that $f_n \to f$ pointwise a.e., and suppose that for each n we have $f_n \le f$ a.e. (however, we do *not* assume that f_n increases monotonically to f). Show that $\int f_n \to \int f$ as $n \to \infty$ (note that these integrals might be ∞).

2. Show that if μ is a finite measure, then convergence in L^∞-norm implies convergence in L^1-norm. Show by example that this can fail if μ is not a finite measure.

3. This problem will prove the *Generalized Dominated Convergence Theorem.* Assume that

(a) $f_n, g_n, f, g \in L^1(X)$,
(b) $f_n \to f$ pointwise a.e.,
(c) $g_n \to g$ pointwise a.e.,
(d) $|f_n| \le g_n$ a.e.,
(e) $\int g_n \to \int g$.

Show that $\int f_n \to \int f$.

Hint: Rework the proof of the DCT.

4. Suppose $f \in L^1(X)$. Given $\varepsilon > 0$, show that there exists a $\delta > 0$ such that

$$A \in \Sigma,\ \mu(A) < \delta \implies \int_A |f| < \varepsilon.$$