MATH 6337 (C. Heil) HOMEWORK #7 Due date: April 19, 2011

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Use Fubini's Theorem to evaluate the integral $\int_0^a \frac{\sin x}{x}\,dx$ (hint: $\int_0^\infty e^{-tx}\,dt = \frac{1}{x}$ when $x > 0$). Then apply the Dominated Convergence Theorem to show that

$$\lim_{a\to\infty} \int_0^a \frac{\sin x}{x}\,dx \;=\; \frac{\pi}{2}.$$

Thus, even though $\frac{\sin x}{x}$ is not Lebesgue integrable on the interval $[0,\infty)$, the improper Riemann integral $\int_0^\infty \frac{\sin x}{x}\,dx$ exists and equals $\frac{\pi}{2}$.

Remark: Complex analysis aficionados may recognize that this problem can also be solved by using contour integration).

2. Let $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ be measurable, and assume that f is a measurable function on $E \times F$. Define $f_x(y) = f(x,y)$, and show that the following two statements are equivalent.

(a) $f = 0$ a.e. on $E \times F$.

(b) For almost every $x \in E$ we have $f_x = 0$ a.e. on F.

3. The *Fourier transform* of a complex-valued function $f \in L^1(\mathbb{R})$ is the function $\hat{f} \colon \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(\xi) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\,dx, \qquad \xi \in \mathbb{R}.$$

Explain why this integral exists for every $\xi \in \mathbb{R}$ (even though f is only defined almost everywhere), and prove that $\hat{f}$ is bounded and uniformly continuous on $\mathbb{R}$.

Hint: To prove continuity, use the DCT.

4. Let $X = Y = [0,1]$, let $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$ (the σ-algebra containing all the Borel subsets of $[0,1]$), let μ be Lebesgue measure on $[0,1]$, and let ν be counting measure on $[0,1]$. Let $D = \{(x,x) : x \in [0,1]\}$ be the diagonal in the unit square $[0,1]^2 = [0,1] \times [0,1]$. Show that

$$\iint \chi_D\, d\mu\, d\nu, \qquad \iint \chi_D\, d\nu\, d\mu, \qquad \iint \chi_D\,(d\mu \times d\mu)$$

are all unequal.

Hint: To compute $\iint \chi_D\,(d\mu \times d\mu) = (\mu \times \nu)(D)$, use the definition of $\mu \times \nu$.