Lusin's Theorem

Theorem 8 (Lusin's Theorem)**.** Given a measurable set $E \subseteq \mathbb{R}^d$ and given $f\colon E \to \mathbb{C}$, the following statements are equivalent.

(a) f is measurable.

(b) For each $\varepsilon > 0$, there exists a closed set $F \subseteq E$ with $|E \setminus F| < \varepsilon$ such that $f|_F$ is continuous, i.e.,

$$\forall\, x_k, x \in F, \quad x_k \to x \implies f(x_k) \to f(x).$$

Proof. (a) $\Rightarrow$ (b). First we prove the result for simple functions. Suppose that $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$ is a simple function, and that the E_j are disjoint. Fix $\varepsilon > 0$. Since E_j is measurable, there exists a closed $F_j \subseteq E_j$ such that

$$|E_j \setminus F_j| \ < \ \frac{\varepsilon}{n}, \quad j = 1, \dots, n.$$

Then

$$F \ = \ \bigcup_{j=1}^{n} E_j$$

is closed, and $|E \setminus F| < \varepsilon$.

If E is a bounded set, then the F_j are compact, and hence

$$\operatorname{dist}(F_j, F_k) \ > \ 0$$

if $j \neq k$. Since ϕ is constant on each F_j, it follows that $\phi|_F$ is continuous.

Exercise: Extend to the case where E is not bounded by considering the sets

$$E_k \ = \ \{x \in E : \|x\| \le k\}.$$

Now let $f\colon E \to \mathbb{C}$ be an arbitrary measurable function. Let ϕ_n be simple functions such that $\phi_n(x) \to f(x)$ for each $x \in E$. Fix $\varepsilon > 0$. By the previous case, for each n we can find a closed $F_n \subseteq E$ such that

$$|E \setminus F_n| \ < \ \frac{\varepsilon}{2^{n+1}}$$

and $\phi_n|_{F_n}$ is continuous.

Suppose that E is bounded. Then by Egoroff's Theorem, there exists a closed $F_0 \subseteq E$ such that

$$|E \setminus F_0| < \frac{\varepsilon}{2}$$

and f_n converges to f uniformly on F_0. Define

$$F \ = \ \bigcap_{n=0}^{\infty} F_n.$$

Then F is closed since each F_n is closed, and

$$|E \setminus F| \ = \ \Big|\bigcup_{n=0}^{\infty} (E \setminus F_n)\Big| \ \le \ \sum_{n=0}^{\infty} |E \setminus F_n| \ \le \ \varepsilon.$$

Since $\phi_n|_{F_n}$ is continuous, $\phi_n|_F$ is continuous as well. And since ϕ_n converges to f uniformly on F, we have that $f|_F$ is continuous. This completes the proof for the case that E is bounded.

Exercise: Extend to the case where E is unbounded by considering the sets

$$E_k \;=\; \{x \in E : k-1 \le \|x\| < k\}.$$

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. By considering the real and imaginary parts of f separately, it suffices to assume that f is real-valued.

By hypothesis, for each $n \in \mathbb{N}$ there exists a closed $F_n \subseteq E$ such that

$$|E \setminus F_n| \;<\; \frac{1}{n}$$

and $f|_{F_n}$ is continuous. Set

$$H \;=\; \bigcup_{n=1}^{\infty} F_n.$$

Then H is an F_σ-set, so is measurable. Also, for every n we have that

$$|E \setminus H| \;\le\; |E \setminus F_n| \;<\; \frac{1}{n}|,$$

so $|E \setminus H| = 0$. Therefore we can write $E = H \cup Z$ where Z has measure zero and is disjoint from H.

If we fix any $a \in \mathbb{R}$, then we have that

$$\begin{aligned}\{f > a\} \;&=\; \{x \in H : f(x) > a\} \,\cup\, \{x \in Z : f(x) > a\} \\ &=\; \bigcup_{n=1}^{\infty} \{x \in F_n : f(x) > a\} \,\cup\, \{x \in Z : f(x) > a\}.\end{aligned}$$

Since each $f|_{F_n}$ is continuous, we have that $\{x \in F_n : f(x) > a\}$ is relatively open with respect to F_n (i.e., it is the intersection of an open set $U \subseteq \mathbb{R}^d$ with F_n) and hence is measurable. And since Lebesgue measure is complete, we know that $\{x \in Z : f(x) > a\}$ is measurable. Therefore we conclude that $\{f > a\}$ is measurable. Since this is true for every real number a, we have shown that f is a measurable function. $\square$