General Notation

Exercises and Problems

The text contains both "Exercises" and "Problems." The exercises are incorporated into the development of the theory in each section. Additional Problems appear at the end of most sections. These problems further develop the material of the section. Hints for selected exercises and problems appear at the end of the volume.

Symbols

We use the symbol $\square$ to denote the end of a proof, and the symbol $\diamondsuit$ to denote the end of a definition, remark, example, or exercise, or the end of the statement of a theorem whose proof will be omitted. An index of mathematical symbols used in the text appears at the end of this volume.

Sets

The empty set is denoted by $\emptyset$.

The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \dots\}$. The set of integers is $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers.

If S is a subset of a set X, then its complement is $X \backslash S = \{x \in X : x \notin S\}$. We sometimes use the abbreviation S^{C} for $X \backslash S$ if the set X is understood.

Sets A and B are *disjoint* if $A \cap B = \emptyset$.

The *symmetric difference* between sets A and B is

$$A \triangle B \;=\; (A \backslash B) \cup (B \backslash A) \;=\; (A \cup B) \setminus (A \cap B).$$

The power set of a set X is the set of all subsets of X, denoted

$$\mathcal{P}(X) \;=\; \bigl\{S : S \subseteq X\bigr\}.$$

The cardinality of X is denoted by $|X|$. A set is countable if it is finite or has the same cardinality as $\mathbb{N}$. For emphasis, we often describe a countable set as being "finite or countably infinite."

Complex Numbers

The *real part* of a complex number $z = a + ib$ $(a,\ b \in \mathbb{R})$ is $\mathrm{Re}(z) = a$, and the *imaginary part* is $\mathrm{Im}(z) = b$. We say that z is *rational* if both its real and imaginary parts are rational numbers. The *complex conjugate* of z is $\bar{z} = a - ib$. The *polar form* of z is $z = re^{i\theta}$ where $r > 0$ and $\theta \in [0, 2\pi)$. The *modulus*, or *absolute value*, of z is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} = r$, and its *argument* is $\arg(z) = \theta$.

The Extended Real Line

The extended real line is

$$\overline{\mathbb{R}} \;=\; \mathbb{R} \cup \{-\infty, \infty\} \;=\; [-\infty, \infty].$$

We use the arithmetic conventions

$$\frac{1}{0} \;=\; \infty, \qquad \frac{1}{\infty} \;=\; 0, \qquad \infty + \infty \;=\; \infty, \qquad 0 \cdot \infty \;=\; 0.$$

However, $\infty - \infty$ is always undefined. The convention $0 \cdot \infty = 0$ may seem odd, but it is the "correct" choice in most equations in analysis.

The infimum and supremum of a set of real numbers $\{a_n\}_{n\in\mathbb{N}}$ always exists in the extended real sense, i.e., $-\infty \le \inf a_n \le \sup a_n \le \infty$. Likewise, if $c_n \ge 0$ for every n then $0 \le \sum_{n=1}^{\infty} c_n \le \infty$ in the sense that the series either converges to a finite value or it diverges to infinity.

If $1 \le p \le \infty$ is given, then its *dual index* is the extended real number p' that satisfies

$$\frac{1}{p} + \frac{1}{p'} \;=\; 1.$$

Explicitly,

$$p' \;=\; \frac{p}{p-1}.$$

The dual index lies in the range $1 \le p' \le \infty$. Some particular dual indices are $1' = \infty$, $2' = 2$, and $\infty' = 1$.

Finite-Dimensional Euclidean Space

We denote the Euclidean norm of a vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ by $|x|$:

$$|x| \;=\; \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$

The closure of $E \subseteq \mathbb{R}^d$ is denoted by $\overline{E}$, and its interior is E°. The *support* of a continuous function f on $\mathbb{R}^d$ is the closure in $\mathbb{R}^d$ of the set $\{x \in \mathbb{R}^d : f(x) \neq 0\}$. Hence a continuous function has compact support if it is zero outside of some bounded set.

Given A, $B \subseteq \mathbb{R}^d$ and $c \in \mathbb{R}$, $x \in \mathbb{R}^d$, we define $A + x = \{a + x : a \in A\}$, $A + B = \{a + b : a \in A,\, b \in B\}$, and $cA = \{ca : a \in A\}$.

Sequences

Given a set X and points $x_i \in X$ for $i \in I$, we let $\{x_i\}_{i \in I}$ denote the sequence indexed by I. We often write $\{x_i\}_{i \in I} \subseteq X$, although it should be noted that $\{x_i\}_{i \in I}$ is a sequence and not just a set. Technically, a sequence $\{x_i\}_{i \in I}$ is shorthand for the mapping $i \mapsto x_i$, and therefore the vectors in a sequence need not be distinct. We generally use the notation $\{x_i\}_{i \in I}$ to denote a sequence of vectors and $(c_i)_{i \in I}$ to denote a sequence of scalars. If the index set I is understood then we may write $\{x_i\}$, $\{x_i\}_i$, (c_i), or $(c_i)_i$. For example, we write "let $\{x_n\}$ be a finite or countable sequence of vectors" to mean that the sequence is either $\{x_n\}_{n=1}^N$ for some $N \in \mathbb{N}$ or or $\{x_n\}_{n \in \mathbb{N}}$.

The *Kronecker delta* is

$$\delta_{ij} \;=\; \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We let δ_n denote the sequence $\delta_n = (\delta_{nk})_{k \in \mathbb{N}}$. That is, the nth component of the sequence δ_n is 1, while all other components are zero. We call δ_n the *nth standard basis vector*, and $\{\delta_n\}_{n \in \mathbb{N}}$ is the *sequence of standard basis vectors*.

Functions

Let X and Y be sets. We write $f \colon X \to Y$ to denote a function with domain X and codomain Y. The *image* or *range* of f is $\mathrm{range}(f) = f(X) = \{f(x) : x \in X\}$. The function f is *injective*, or *1-1*, if $f(a) = f(b)$ implies $a = b$. It is *surjective*, or *onto*, if $f(X) = Y$. It is *bijective* if it is both injective and surjective. If S is a subset of X, then the restriction of f to the domain S is denoted by $f|_S$.

Given $A \subseteq X$, the *direct image* of A under f is

$$f(A) \;=\; \{f(x) : x \in A\}.$$

If $B \subseteq Y$, then the *inverse image* of B under f is

$$f^{-1}(B) \;=\; \{x \in X : f(x) \in B\}.$$

Given a real-valued function f, we set

$$f^+(x) \;=\; \max\{f(x), 0\} \qquad \text{and} \qquad f^-(x) \;=\; \max\{-f(x), 0\}.$$

We have the relations

$$f(x) = f^+(x) - f^-(x) \qquad \text{and} \qquad |f(x)| = f^+(x) + f^-(x).$$

We call f^+ the *positive part* and f^- the *negative part* of f.

A real-valued or complex-valued function f is *bounded* on $E \subseteq X$ if there exists a real number M such that $|f(x)| \le M$ for every $x \in E$.

Given a set X, the *characteristic function* of a subset $A \subseteq X$ is the function $\chi_A \colon X \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Given $E \subseteq \mathbb{R}$, a function $f \colon E \to \mathbb{R}$ is *monotone increasing* on E if

$$\forall\, x, y \in E, \quad x < y \implies f(x) \le f(y),$$

and it is *strictly increasing* on E if

$$\forall\, x, y \in E, \quad x < y \implies f(x) < f(y).$$

Extended Real-Valued Functions

Let $f \colon X \to \overline{\mathbb{R}}$ be an extended real-valued function on a set X. There are several convenient shorthand notations that we will use to denote certain inverse images.

To avoid multiplicities of parentheses, brackets, and braces, we write

$$\begin{aligned} f^{-1}(a,b) &= f^{-1}((a,b)), \\ f^{-1}[a,\infty) &= f^{-1}([a,\infty)), \\ f^{-1}\{x\} &= f^{-1}(\{x\}), \end{aligned}$$

and so forth. We also use the following shorthands:

$$\begin{aligned} \{f > a\} &= \{x \in X : f(x) > a\} = f^{-1}(a,\infty], \\ \{f \ge a\} &= \{x \in X : f(x) \ge a\} = f^{-1}[a,\infty], \\ \{f = a\} &= \{x \in X : f(x) = a\} = f^{-1}\{a\}, \\ \{a < f < b\} &= \{x \in X : a < f(x) < b\} = f^{-1}(a,b), \end{aligned}$$

and other variations on this theme. We use similar notations involving two functions f and g, such as

$$\begin{aligned} \{f \ge g\} &= \{x \in X : f(x) \ge g(x)\}, \\ \{f = g\} &= \{x \in X : f(x) = g(x)\}. \end{aligned}$$

Increasing Sequences of Functions

If $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of extended real-valued functions on a set X such that

$$f_1(x) \le f_2(x) \le \cdots \qquad \text{for all } x \in X,$$

then

$$f(x) = \lim_{k\to\infty} f_k(x)$$

exists for each x in the extended real sense. In this case we say that $\{f_k\}_{k\in\mathbb{N}}$ *is an increasing sequence of functions* and f_k *increases pointwise to* f. We write

$$f_k \nearrow f$$

to mean that the functions f_k increase pointwise to f.