0.1 Pointwise Convergence

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions on a set X, either complex-valued or extended real-valued. We say that f_k *converges pointwise* to a function f if for each *individual* element $x \in X$, the scalar $f_k(x)$ converges to $f(x)$ as $k \to \infty$. We state this explicitly as a definition.

Definition 0.5 (Pointwise Convergence). Let f_k, f be extended real-valued or complex-valued functions defined on a set X.

(a) We say that f_k *converges pointwise to* f, and write $f_k \to f$ *pointwise*, if

$$\forall\, x \in X, \quad \lim_{k\to\infty} f_k(x) \;=\; f(x).$$

(b) If $f_k \to f$ pointwise and the f_k are extended real-valued functions that are monotone increasing, i.e.,

$$f_1(x) \;\le\; f_2(x) \;\le\; \cdots, \qquad \text{all } x \in X,$$

then we say that f_k *increases pointwise to* f. We denote this with the notation $f_k \nearrow f$, and make a similar definition for monotone decreasing sequences. ◊

If the functions f_k are complex-valued then we can rewrite the definition of pointwise convergence in a form that will allow a better comparison to the coming discussions of uniform convergence and convergence in measure. Specifically, if $f_k(x)$ is a finite scalar (real or complex) for each x, then f_k converges pointwise to f if and only if

$$\forall\, x \in X, \quad \lim_{k\to\infty} |f(x) - f_k(x)| \;=\; 0. \tag{0.1}$$

It is instructive for the reader to write this out completely in terms of epsilons.

Example 0.6. As an example, consider $f_k = \chi_{(0,\frac{1}{k})}$, the characteristic function of the open interval $(0, \frac{1}{k})$. For each individual $x \in \mathbb{R}$, we have

$$\lim_{k\to\infty} |\chi_{(0,\frac{1}{k})}(x) - 0| \;=\; 0.$$

Hence $\chi_{(0,\frac{1}{k})}$ converges pointwise to the zero function (in fact, $\chi_{(0,\frac{1}{k})} \searrow 0$). However, if $0 < x < 1$ then the smaller that x is, the longer we have to wait (more precisely, the larger that k must become) in order for $\chi_{(0,\frac{1}{k})}(x)$ to get close to 0. ◊

A loose summary of pointwise convergence is that $f_k(x)$ and $f(x)$ must be close together for all k large enough, but how long we have to wait for $f_k(x)$ and $f(x)$ to get close can depend on which element x that we are looking at.

Additional Problems

0.5. Let $\mathcal{F}_b(\mathbb{R})$ be the vector space of all bounded, complex-valued functions on $\mathbb{R}$. Show that there is no norm $\|\cdot\|$ on $\mathcal{F}_b(\mathbb{R})$ such that

$$f_k \to f \text{ pointwise} \quad \iff \quad \lim_{k\to\infty} \|f - f_k\| \;=\; 0.$$

0.2 Uniform Convergence

Again let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions on a set X. Uniform convergence requires a "simultaneity" of convergence that is not required for pointwise convergence. In essence, we require that $f_k(x)$ and $f(x)$ be simultaneously close for all $x \in X$ when k is large enough, or we might describe this by saying that the *maximum difference* between $f_k(x)$ and $f(x)$ must be small for all large enough k (technically, it would be better to speak of the *suprema of the differences* instead of the *maximum difference*, but the latter terminology seems much more descriptive).

Note that in order to for the "maximum difference" between functions to be small, the function values cannot be $\pm\infty$. Therefore, when speaking of uniform convergence we do not allow our functions take extended real values. Since real-valued functions are just a special case of complex-valued functions, this means that we can state the definition for complex-valued functions. There are several equivalent ways to precisely formulate the definition of uniform convergence; we choose to do this as follows.

Definition 0.7 (Uniform Convergence). Let f_k, f be complex-valued functions defined on a set X. We say that f_k *converges uniformly to* f, and write $f_k \to f$ *uniformly*, if

$$\lim_{k\to\infty} \Bigl(\sup_{x\in X} |f(x) - f_k(x)|\Bigr) \;=\; 0. \qquad \diamondsuit \tag{0.2}$$

In precise terms, the "maximum difference" between f_k and f that we have been loosely referring to is the number $\sup_{x\in X} |f(x) - f_k(x)|$. Because this quantity is so important, we introduce the following terminology.

Definition 0.8 (Uniform Norm). Let X be a set.

(a) The *uniform norm* of a function $f\colon X \to \mathbb{C}$ is the supremum of $|f|$ over X, and is denoted by

$$\|f\|_{\mathrm{u}} \;=\; \sup_{x\in X} |f(x)|.$$

(b) If f and g are two functions on X, then $\|f - g\|_{\mathrm{u}}$ is called the *distance between f and g in the uniform norm.* $\diamondsuit$

In this terminology, uniform convergence simply means that the distance between f_k and f shrinks to zero when measured by the uniform norm:

$$f_k \to f \text{ uniformly} \quad \iff \quad \lim_{k\to\infty} \|f - f_k\|_{\mathrm{u}} \;=\; 0. \tag{0.3}$$

We can literally think of $\|f - f_k\|_{\mathrm{u}}$ as being the distance from f_k to f. If this distance converges to zero as k increases, then f_k converges to f uniformly.

Uniform convergence implies pointwise convergence, but the next example shows that the converse fails in general.

Example 0.9. We saw in Example 0.6 that the functions $\chi_{(0,\frac{1}{k})}$ converge pointwise to the zero function. However, since

$$\sup_{x\in\mathbb{R}} |\chi_{(0,\frac{1}{k})}(x) - 0| \;=\; 1$$

for every k, the maximum difference between $\chi_{(0,\frac{1}{k})}$ and the zero function does not converge to zero. Hence $\chi_{(0,\frac{1}{k})}$ does not converge uniformly to the zero function. $\diamondsuit$

The next exercise gives four properties that are satisfied by the uniform norm. In general, a *norm* is a function $\|\cdot\|$ on a vector space V that satisfies the analogues of these four properties on the space V. The fact that there is a norm lying behind the notion of uniform convergence makes this type of convergence very easy to deal with in certain ways. In contrast, pointwise convergence cannot be characterized in terms of a single norm (Problem 0.5).

Exercise 0.10. Let $\mathcal{F}_b(X)$ (b for "bounded") be the set of all bounded, complex-valued functions on X, i.e.,

$$\mathcal{F}_b(X) \;=\; \big\{f\colon X \to \mathbb{C} \,:\, f \text{ is bounded}\big\}.$$

This is a vector space over the complex field. Show that uniform norm satisfies the following properties on $\mathcal{F}_b(X)$:

(a) $0 \le \|f\|_{\mathrm{u}} \le \infty$ for all $f \in \mathcal{F}_b(X)$,

(b) $\|f\|_{\mathrm{u}} = 0$ if and only if $f = 0$ (the zero function),

(c) $\|cf\|_{\mathrm{u}} = |c|\,\|f\|_{\mathrm{u}}$ for all $f \in \mathcal{F}_b(X)$ and scalars $c \in \mathbb{C}$,

(d) $\|f+g\|_{\mathrm{u}} \le \|f\|_{\mathrm{u}} + \|g\|_{\mathrm{u}}$ for all $f,\, g \in \mathcal{F}_b(X)$. $\diamondsuit$

Property (b) in the list above is a *uniqueness* requirement: Only the zero vector can have zero norm. Property (c) is sometimes called the *homogeneity* or *positive homogeneity* property of the uniform norm, and Property (d) is usually referred to as the *Triangle Inequality* for the uniform norm. If we replace $\mathcal{F}_b(X)$ by a different vector space V, then properties (a)–(d) above are exactly what are required in order for a function on V to be called a norm.

As we stated in equation (0.3), uniform convergence means that the distance between f_k and f shrinks as k increases. Consequently, f_k is close to f

when k is large enough, say within ε in distance for all $k > N$. Since the Triangle Inequality implies that

$$\|f_j - f_k\|_u \;\le\; \|f_j - f\| + \|f - f_k\|_u, \tag{0.4}$$

it follows that f_j must be close to f_k for all j, k large enough (specifically, the distance will be at most 2ε for all j, $k > N$). Therefore, a sequence that converges uniformly is *Cauchy* in the following sense.

Definition 0.11 (Uniformly Cauchy). Let X be a set, and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of funtions in $\mathcal{F}_b(X)$. We say that $\{f_k\}_{k\in\mathbb{N}}$ is *uniformly Cauchy* or *Cauchy in the uniform norm* if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, j, k \ge N, \quad \|f_j - f_k\|_u < \varepsilon. \qquad \diamondsuit \tag{0.5}$$

Sometimes we write $\lim_{j,k\to\infty} \|f_j - f_k\|_u = 0$ as an abbreviation for equation (0.5), but it should be understood that this is not a limit in the usual sense. Instead, it is only a shorthand for equation (0.5).

It follows from equation (0.4) that every uniformly convergent sequence in $\mathcal{F}_b(X)$ is uniformly Cauchy. The converse question is extremely important. Indeed, this is a fundamental question whenever we have a norm on a vector space: If $\{f_k\}$ is a Cauchy sequence, must it converge to some vector in the space? If the answer is yes, then we say that the space is *complete*. For example, $\mathbb{R}^d$ and $\mathbb{C}^d$ are complete with respect to the usual Euclidean norm. We show next that $\mathcal{F}_b(X)$ is complete with respect to the uniform norm.

Theorem 0.12. *If X is a set, then the space $\mathcal{F}_b(X)$ is complete with respect to the uniform norm. That is, if $\{f_k\}_{k\in\mathbb{N}}$ is a uniformly Cauchy sequence in $\mathcal{F}_b(X)$, then there exists a function $f \in \mathcal{F}_b(X)$ such that $f_k \to f$ uniformly.*

Proof. Suppose that $\{f_k\}_{k\in\mathbb{N}}$ is a uniformly Cauchy sequence in $\mathcal{F}_b(X)$. If we fix an element $x \in X$, then it follows from the definition of the uniform norm that

$$|f_j(x) - f_k(x)| \;\le\; \|f_j - f_k\|_u, \qquad \text{all } j, k \in \mathbb{N}.$$

Consequently $\{f_k(x)\}_{k\in\mathbb{N}}$ is a Cauchy sequence of complex scalars. Since $\mathbb{C}$ is complete with respect to the Euclidean norm, this sequence of scalars must converge to some complex number. Therefore we can define

$$f(x) \;=\; \lim_{k\to\infty} f_k(x).$$

Doing this for each $x \in X$ gives us a function f such that $f_k \to f$ *pointwise.* We still have to show that f belongs to the space $\mathcal{F}_b(X)$, and further we have to show that $f_k \to f$ *uniformly.*

Fix an $\varepsilon > 0$. By definition of Cauchy sequence, there must exist an N such that $\|f_j - f_k\|_u < \varepsilon$ for all j, $k > N$. If we fix an integer $k > N$, then for every $x \in X$ we have

$$|f(x) - f_k(x)| \;=\; \lim_{j\to\infty} |f_j(x) - f_k(x)| \;\le\; \|f_j - f_k\|_{\mathrm{u}} \;\le\; \varepsilon.$$

Taking the supremum over all $x \in X$, we see that $\|f - f_k\|_{\mathrm{u}} \le \varepsilon$ for all $k > N$. This precisely says that f_k converges uniformly to f. Finally, by the Triangle Inequality, for any k we have

$$\|f\|_{\mathrm{u}} \;\le\; \|f - f_k\|_{\mathrm{u}} + \|f_k\|_{\mathrm{u}} \;<\; \infty,$$

so f is bounded and hence belongs to $\mathcal{F}_b(X)$. □

A complete normed vector space is also called a *Banach space* (see Definition 0.4), so we have shown that $\mathcal{F}_b(X)$ is a Banach space with respect to the uniform norm.

Remark 0.13. The reader should be aware that the word "complete" is heavily overused in mathematics. We have already assigned one meaning to this word when we defined a *complete measure*, and now we have used the same word to describe the entirely distinct notion of a *complete space*. As long as we are careful to observe the context in which the word "complete" is being used, this will not be a problem. ♢