1

Lebesgue Measure

There are two glasses of water on my desk. The little glass contains v_1 units of water and the big glass has v_2 units. I pour the two glasses into a single larger glass, which now contains $v_1 + v_2$ units of water. Volume of incompressible fluids appears to be additive. It seems obvious that I should be able to put this on a precise mathematical footing and find a function $|\cdot|$ that assigns to every set $E \subseteq \mathbb{R}^d$ a "volume" or "measure" $|E|$ in such that way that:

(a) $0 \le |E| \le \infty$,

(b) if $E_1, E_2, \dots$ are finitely or countably many *disjoint* sets then

$$\Bigl|\bigcup_n E_n\Bigr| = \sum_n |E_n|,$$

(c) $|E + x| = |E|$ for all $x \in \mathbb{R}^d$, and

(d) the unit cube $[0,1)^d$ has unit measure, i.e., $\bigl|[0,1)^d\bigr| = 1$.

The Axiom of Choice implies that *no such function exists!* We will prove this in Section 1.7. Fortunately, this turns out to be only a small stumbling block. The Axiom of Choice implies that bizarre *nonmeasurable sets* exist, but if we are careful we can solve this difficulty by working with the class of well-behaved *measurable sets*. Instead of trying to measure *everything*, we only try to measure the "good sets." Moreover, the sets that we encounter in practice are almost always "good sets."

Here is the plan for this chapter.

(1) We start with a basic class of subsets of $\mathbb{R}^d$ that we know how we want to measure. Among the simplest choices are balls and rectangular boxes, and we choose to start with boxes (because they are simpler in certain ways, e.g., we can tile $\mathbb{R}^d$ with boxes but not with balls). We declare that the measure of a rectangular box is its volume.

(2) Next, we find a way to extend the notion of volume of boxes to generic subsets of $\mathbb{R}^d$. For every set $E \subseteq \mathbb{R}^d$ we define a nonnegative, extended real-value number $|E|_e$, and we do this in a way that naturally extends the notion of volume of boxes. This number $|E|_e$ is the *exterior Lebesgue measure of E*. The good news is that every subset of $\mathbb{R}^d$ has a uniquely defined exterior measure. The bad—and rather unsettling—news is that requirement (b) above does *not* hold for exterior Lebesgue measure in general. In fact, the Axiom of Choice implies that there exist *disjoint* sets $A, B \subseteq \mathbb{R}^d$ such that $|A \cup B|_e < |A|_e + |B|_e$ (see Example 1.50).

(3) Finally, we find a way to restrict our attention to a smaller class of sets $\mathcal{L}_{\mathbb{R}^d}$, the *Lebesgue measurable subsets of* $\mathbb{R}^d$, in such a way that $|\cdot|_e$ restricted to $\mathcal{L}_{\mathbb{R}^d}$ satisfies requirements (a)–(d) above—not for *all* subsets of of $\mathbb{R}^d$ but at least for the Lebesgue measurable subsets. This is *Lebesgue measure on* $\mathbb{R}^d$.

In summary, Lebesgue measure is the extension of volume from boxes to a large class of subsets of $\mathbb{R}^d$ on which requirements (a)–(d) are satisfied. However, while Lebesgue measure is the prototypical example of a measure on $\mathbb{R}^d$, it is not the only way to measure things. For example, in some applications it may be more important to know *where* a set is than how much "volume" it contains, and so we might define the "big sets" to be the ones that contain a certain point. This idea leads to the δ-*measure* on $\mathbb{R}^d$, defined on sets $E \subseteq \mathbb{R}^d$ by $\delta(E) = 1$ if E contains the origin and $\delta(E) = 0$ otherwise. This turns out to be a perfectly good "measure" on $\mathbb{R}^d$, and there are many other measures on $\mathbb{R}^d$ that appear naturally in other contexts. These measures need not satisfy all of requirements (a)–(d) above—we may give up the idea of "volume" as the ideal meaning of "measure," and instead focus on other properties as being central to the idea of measure. To be precise, here is the list of properties that we require a function to satisfy in order to be called a *measure* on a set X.

Definition 1.1 (Measure). Let X be a set.

(a) A σ-*algebra* or σ-*field* on X is a nonempty collection Σ of subsets of X such that:

- Σ is closed under countable unions:

$$E_1, E_2, \ldots \in \Sigma \implies \bigcup_k E_k \in \Sigma,$$

- Σ is closed under complements:

$$E \in \Sigma \implies E^{\mathrm{C}} = X \backslash E \in \Sigma.$$

(b) A function $\mu\colon \Sigma \to [0, \infty]$ is a *measure* on X with respect to a σ-algebra Σ if:

- $\mu(\emptyset) = 0$, and

- μ is *countably additive*:

$$E_1, E_2, \ldots \in \Sigma \text{ are disjoint} \quad \Longrightarrow \quad \mu\Big(\bigcup_k E_k\Big) \;=\; \sum_k \mu(E_k).$$

If μ is a measure on X with respect to a σ-algebra Σ, then we say that (X, Σ, μ) is a *measure space*. $\diamondsuit$

In short, the σ-algebra Σ is the collection of "good subsets" of X that we are allowed to measure, and the function μ tells us how to measure them. For Lebesgue measure, the good sets will be the Lebesgue measurable subsets of $\mathbb{R}^d$, while other measures may employ different σ-algebras of good sets. Here in Chapter 1 we will focus on the process of constructing Lebesgue measure. We will create exterior Lebesgue measure and explore its properties, and then find the σ-algebra of Lebesgue measurable sets. In Chapter 2 we turn to generic measures on sets, and the later chapters then use the machinery of measure theory to create and apply a theory of integration.

1.1 Exterior Lebesgue Measure

Lebesgue measure is predicated on the idea of extending the notion of volume from "simple" sets to more complicated ones. We begin with a class of objects that we know how to measure—the rectangular parallelepipeds in $\mathbb{R}^d$ whose sides are parallel to the coordinate axes. For simplicity we refer to these as "boxes."

Definition 1.2. A *box* in $\mathbb{R}^d$ is a set of the form

$$Q \;=\; [a_1, b_1] \times \cdots \times [a_d, b_d] \;=\; \prod_{i=1}^{d} [a_i, b_i]. \tag{1.1}$$

The *volume* of this box is

$$\mathrm{vol}(Q) \;=\; (b_1 - a_1) \cdots (b_d - a_d) \;=\; \prod_{i=1}^{d} (b_i - a_i). \qquad \diamondsuit$$

Whenever we write "box" without qualification, we mean a closed rectangular box of the type defined in equation (1.1). If $d = 1$ then boxes are closed intervals and their volume is their length, while if $d = 2$ then boxes are closed rectangles and their volume is their area.

In order to measure generic sets, we approximate them by things we know. If we cover a set E by boxes then the sum of the volumes of those boxes should exceed the "measure" of E, whatever that is. Some coverings will do a better job of approximating E, but each covering will be a little "too big." We define the exterior Lebesgue measure of E to be the infimum of these sums of volumes over all possible coverings of E by boxes.

Notation 1.3. (a) As stated in the opening section on General Notation, we often write $\{Q_k\}$ to denote a collection where k runs through some implicit index set (usually countable).

(b) Given a set $E \subseteq \mathbb{R}^d$, if $\{Q_k\}$ is a countable collection of boxes such that $E \subseteq \cup Q_k$ then we say that $\{Q_k\}$ *is a countable cover of* E *by boxes.* $\diamondsuit$

Definition 1.4. The *exterior Lebesgue measure* or *outer Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e \;=\; \inf\biggl\{\sum_k \mathrm{vol}(Q_k)\biggr\},$$

where the infimum is taken over all *finite or countably infinite* collections of boxes $\{Q_k\}$ such that $E \subseteq \cup Q_k$. $\diamondsuit$

Every subset E of $\mathbb{R}^d$ has a uniquely defined exterior measure that lies in the range $0 \le |E|_e \le \infty$. For example, Problem 1.1 asks for a proof that

$$|\emptyset|_e \;=\; 0 \qquad \text{and} \qquad |\mathbb{R}^d|_e \;=\; \infty.$$

However, despite its name, "exterior Lebesgue measure" is *not* a measure in the sense of Definition 1.1. In general, countable additivity fails for exterior Lebesgue measure (we will prove this in Section 1.7).

The following are immediate, but very important, consequences of the definition of exterior Lebesgue measure.

Exercise 1.5. Given $E \subseteq \mathbb{R}^d$, prove the following statements.

(a) If $\{Q_k\}$ is *any* countable cover of E by boxes, then

$$|E|_e \;\le\; \sum \mathrm{vol}(Q_k).$$

(b) Given $\varepsilon > 0$, there exists *some* countable cover of E by boxes Q_k such that

$$|E|_e \;\le\; \sum_k \mathrm{vol}(Q_k) \;\le\; |E|_e + \varepsilon.$$

Note that we might have $|E|_e = \infty$ in the line above.

(c) A translation $Q + h = \{x + h : x \in Q\}$ of a box Q is another box with the same volume. As a consequence, exterior Lebesgue measure is *translation-invariant*:

$$|E + h|_e \;=\; |E|_e \qquad \text{for all } h \in \mathbb{R}^d.$$

(d) By definition, if E is a bounded subset of $\mathbb{R}^d$ then it is contained in some box Q. Therefore $|E|_e \le \mathrm{vol}(Q) < \infty$. On the other hand, there exist unbounded sets that have positive but finite exterior measure. $\diamondsuit$

Here are some examples of exterior measures of sets.

Example 1.6. Suppose that $E = \{x_k\}$ is a countable subset of $\mathbb{R}^d$, and fix $\varepsilon > 0$. For each k, choose a box Q_k that contains the point x_k and has volume $\mathrm{vol}(Q_k) < 2^{-k}\varepsilon$. Then $E \subseteq \cup Q_k$, so $|E|_e \le \sum \mathrm{vol}(Q_k) \le \varepsilon$. Since ε is arbitrary, we conclude that $|E|_e = 0$. Thus, every countable subset of $\mathbb{R}^d$ has zero exterior Lebesgue measure.

In particular, the set of rationals $\mathbb{Q}$ is a countable subset of $\mathbb{R}$, so $|\mathbb{Q}|_e = 0$. Thus $\mathbb{Q}$ is a "very small" part of $\mathbb{R}$ in a measure-theoretic sense. This contrasts with the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and therefore is a "very large" part of $\mathbb{R}$ in a topological sense. A set and its closure can have very different exterior measures! ◇

Example 1.7. Consider the boundary of the unit square $Q = [0,1]^2$ in $\mathbb{R}^2$. The boundary is a union of four line segments ℓ_1, ℓ_2, ℓ_3, ℓ_4. Each line segment is an uncountable set, but (as a subset of $\mathbb{R}^2$) it has measure zero since we can cover it with a single box that has arbitrarily small area. For example, for the bottom line segment ℓ_1 we can write

$$\ell_1 \;=\; \bigl\{(x,0) : 0 \le x \le 1\bigr\} \;\subseteq\; [0,1] \times [-\varepsilon, \varepsilon] \;=\; Q_\varepsilon,$$

so $|\ell_1|_e \le \mathrm{vol}(Q_\varepsilon) = 2\varepsilon$. Therefore the two-dimensional exterior Lebesgue measure of the line segment ℓ_1 is zero. Similarly, ∂Q can be covered by four boxes with arbitrarily small volume, so $|\partial Q|_e = 0$, and an extension of this argument shows that the boundary of every box in $\mathbb{R}^d$ has exterior measure zero. ◇

So, we have shown that the boundary of a box has exterior measure zero. This raises an interesting question—is it true that the boundary of every closed subset of $\mathbb{R}^d$ has measure zero? We will answer this question shortly, but give it some thought before reading onward!

Now, a box in $\mathbb{R}$ is just an interval, so its boundary is the finite set consisting of the two endpoints of the interval. However, if $d \ge 2$ then the boundary of a box is not merely an infinite set but is actually *uncountable*—yet even so it has measure zero. Therefore, at least when $d \ge 2$, there exist uncountable subsets of $\mathbb{R}^d$ that have measure zero. It is not quite so obvious whether there exist uncountable subsets of $\mathbb{R}$ that have zero exterior measure, but we will construct one next.

Example 1.8 (The Cantor Set). Define

$$\begin{aligned}
F_0 &= [0,1], \\
F_1 &= \bigl[0,\tfrac{1}{3}\bigr] \cup \bigl[\tfrac{2}{3},1\bigr], \\
F_2 &= \bigl[0,\tfrac{1}{9}\bigr] \cup \bigl[\tfrac{2}{9},\tfrac{1}{3}\bigr] \cup \bigl[\tfrac{2}{3},\tfrac{7}{9}\bigr] \cup \bigl[\tfrac{8}{9},1\bigr], \\
&\;\;\vdots
\end{aligned}$$

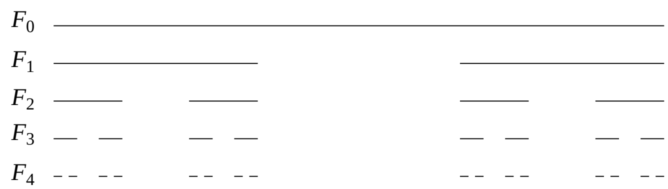

Fig. 1.1. The Cantor set C is the intersection of the sets F_n over all $n \in \mathbb{N}$.

(see Figure 1.1). For a given n, F_n is a closed set that consists of 2^n intervals, each with length 3^{-n}. Therefore, since boxes in one dimension are just closed intervals, the set F_n is covered by 2^n boxes each with volume 3^{-n}. Exercise 1.5(a) therefore tells us that

$$0 \le |F_n|_e \le 2^n \, 3^{-n} = (2/3)^n.$$

(In fact, the exterior measure of F_n is precisely $(2/3)^n$, but we have not proved this fact yet and will not need to use it here.) We create the set F_{n+1} by removing the middle third from each of the intervals that comprise F_n. The classical "middle-thirds" *Cantor set* is $C = \cap F_n$ The Cantor set is closed because it is an intersection of closed sets. Moreover $C \subseteq F_n$, so C is covered by the 2^n intervals that comprise F_n. Applying Exercise 1.5(a), it follows that

$$0 \le |C|_e \le (2/3)^n.$$

This is true for every integer $n \ge 0$, so the exterior Lebesgue measure of C is precisely $|C|_e = 0$. Even so, Problem 1.7 sketches a direct proof that the Cantor set is uncountable. ♢

The Cantor set has many remarkable properties. For example, C contains no interior points and equals its own boundary (Problem 1.7). Moreover, C is a perfect set in the sense of the following definition.

Definition 1.9. Given $S \subseteq \mathbb{R}^d$, let S' be the set of all accumulation points of S. We say that S is *perfect* if S is nonempty and $S = S'$. ♢

Problem 1.9 asks for a proof that every perfect set is uncountable. Although we will not prove it, the *Cantor–Bendixson Theorem* is a fundamental result that states that every nonempty closed set $F \subseteq \mathbb{R}^d$ can be uniquely written as $F = E \cup Z$ where E is perfect and Z is countable.

Remark 1.10. Some variations on the construction of the Cantor set are given in Problems 1.8 and 1.25. In particular, Problem 1.25 shows how to construct a Cantor-like set P that has positive measure. This set P is closed and equals its own boundary, so—despite our intuition that this should be impossible—*there exists a closed set whose boundary has positive exterior measure!* ♢

Now we explore some of the basic properties of exterior Lebesgue measure.

Lemma 1.11 (Monotonicity). *If* $A \subseteq B \subseteq \mathbb{R}^d$, *then* $|A|_e \le |B|_e$.

Proof. If $\{Q_k\}$ is a countable cover of B by boxes, then it is also a cover of A by boxes, so

$$|A|_e \;\le\; \sum_k \operatorname{vol}(Q_k).$$

This is true for every possible covering of B, so

$$|A|_e \;\le\; \inf\left\{\sum_k \operatorname{vol}(Q_k):\ \text{all countable covers of } B \text{ by boxes}\right\} \;=\; |B|_e. \quad \square$$

Since a box Q can be covered by a collection containing a single box (itself), it follows from Definition 1.2 that $|Q|_e \le \operatorname{vol}(Q)$. However, it requires some care to show that the exterior measure of a box actually coincides with its volume. In order to prove this, we will need the facts about volumes of boxes given in the following exercise.

Exercise 1.12. Given two boxes Q and K in $\mathbb{R}^d$ whose interiors intersect, observe that:

(a) $Q \cap K$ is a box, and

(b) $Q \backslash K$ can be written as the union of finitely many boxes that only intersect along their boundaries.

Use these facts to show that if a box Q is covered by finitely many boxes $Q_1, \dots, Q_N$, then

$$\operatorname{vol}(Q) \;\le\; \sum_{k=1}^{N} \operatorname{vol}(Q_k). \qquad \diamondsuit \tag{1.2}$$

Theorem 1.13 (Consistency with Volume). *If Q is a box in* $\mathbb{R}^d$, *then* $|Q|_e = \operatorname{vol}(Q)$.

Proof. Since we already know that $|Q|_e \le \operatorname{vol}(Q)$, our task is to prove the opposite inequality. Let $\{Q_k\}$ be a countable covering of Q by boxes, and fix $\varepsilon > 0$. For each k, let Q_k^* be any box such that:

- Q_k is contained in the interior of Q_k^*, i.e., $Q_k \subseteq (Q_k^*)^\circ$, and
- $\operatorname{vol}(Q_k^*) \le (1+\varepsilon)\operatorname{vol}(Q_k)$.

Then

$$Q \;\subseteq\; \bigcup_k Q_k \;\subseteq\; \bigcup_k (Q_k^*)^\circ,$$

so $\{(Q_k^*)^\circ\}$ is a countable open cover of Q. As Q is compact, this open cover must have a finite subcover. That is, there is some integer $N > 0$ such that

$$Q \subseteq \bigcup_{k=1}^{N} (Q_k^*)^\circ \subseteq \bigcup_{k=1}^{N} Q_k^*.$$

Applying Exercise 1.12, we obtain the following fact about volumes:

$$\mathrm{vol}(Q) \leq \sum_{k=1}^{N} \mathrm{vol}(Q_k^*) \leq (1+\varepsilon) \sum_{k=1}^{N} \mathrm{vol}(Q_k) \leq (1+\varepsilon) \sum_{k} \mathrm{vol}(Q_k).$$

Since this is true for every covering $\{Q_k\}_k$, it follows from the definition of $|Q|_e$ that

$$\mathrm{vol}(Q) \leq (1+\varepsilon) \inf\biggl\{\sum_{k} \mathrm{vol}(Q_k)\biggr\} = (1+\varepsilon)\, |Q|_e.$$

However, ε is arbitrary, so $\mathrm{vol}(Q) \leq |Q|_e$. □

Next we prove an important property of exterior Lebesgue measure called *countable subadditivity*, which states that the exterior measure of a countable union of sets is no more than the sum of the measures of the sets. Note that we are *not* requiring the sets here to be disjoint.

Theorem 1.14 (Countable Subadditivity). *If* $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, *then*

$$\biggl|\bigcup_{k=1}^{\infty} E_k\biggr|_e \leq \sum_{k} |E_k|_e.$$

Proof. If any E_k satisfies $|E_k|_e = \infty$ then we are done, so let us assume that $|E_k|_e < \infty$ for every k. Fix any $\varepsilon > 0$. Then for each $k \in \mathbb{N}$ we can find a covering $\{Q_j^{(k)}\}_j$ of E_k by countably many boxes $Q_j^{(k)}$ in such a way that

$$\sum_{j} \mathrm{vol}\bigl(Q_j^{(k)}\bigr) \leq |E_k|_e + \frac{\varepsilon}{2^k}. \tag{1.3}$$

Then $\{Q_j^{(k)}\}_{j,k}$ is a covering of $\cup E_k$ by countably many boxes:

$$\bigcup_{k} E_k \subseteq \bigcup_{k,j} Q_j^{(k)}.$$

Therefore

$$\begin{aligned}
\biggl|\bigcup_{k} E_k\biggr|_e &\leq \sum_{k,j} \mathrm{vol}\bigl(Q_j^{(k)}\bigr) && \text{(by Definition 1.4)} \\
&\leq \sum_{k} \Bigl(|E_k|_e + \frac{\varepsilon}{2^k}\Bigr) && \text{(by equation (1.3))} \\
&= \biggl(\sum_{k} |E_k|_e\biggr) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, the result follows. □

Remark 1.15. (a) The proofs of Theorems 1.13 and 1.14 illustrate two useful ways of "getting within ε" when dealing with countable sums. In the proof of Theorem 1.13 we introduced a multiplicative $1+\varepsilon$ factor by constructing Q_k^* so that $\mathrm{vol}(Q_k^*) \le (1+\varepsilon)\,\mathrm{vol}(Q_k)$. In contrast, in the proof of Theorem 1.14 we introduced an additive term of the form $2^{-k}\varepsilon$ in equation (1.3). These two techniques will be employed often in future proofs.

(b) Although we stated Theorem 1.14 for countably infinite collections of sets $E_1, E_2, \dots$, it implicitly applies to finite collections as well. If we have only finitely many sets $E_1, \dots, E_N$ then we can simply define $E_k = \emptyset$ for $k > N$ and apply Theorem 1.14 to the sequence $\{E_k\}_{k\in\mathbb{N}}$ to obtain

$$\Bigl|\bigcup_{k=1}^{N} E_k\Bigr|_e \;=\; \Bigl|\bigcup_{k=1}^{\infty} E_k\Bigr|_e \;\le\; \sum_{k=1}^{\infty} |E_k|_e \;\le\; \sum_{k=1}^{N} |E_k|_e.$$

By applying the same trick, other theorems that are stated for countably infinite collections usually apply to finite collections as well.

(c) Subadditivity need not hold when dealing with *uncountable* collections of sets. For example, we can write the real line as an uncountable union of singletons:

$$\mathbb{R} \;=\; \bigcup_{x\in\mathbb{R}} \{x\}.$$

However $|\{x\}|_e = 0$, so

$$|\mathbb{R}|_e = \infty \qquad \text{yet} \qquad \sum_{x\in\mathbb{R}} |\{x\}|_e \;=\; 0. \qquad \diamondsuit$$

Applying Theorem 1.14 to sets that have measure zero, we see that *the countable union of sets with measure zero has measure zero.*

Corollary 1.16. *If $Z_k \subseteq \mathbb{R}^d$ and $|Z_k|_e = 0$ for each $k \in \mathbb{N}$, then*

$$\Bigl|\bigcup_{k=1}^{\infty} Z_k\Bigr|_e \;=\; 0. \qquad \diamondsuit$$

Our final theorem in this section gives a type of "regularity" property for exterior Lebesgue measure: Every set E can be surrounded by an *open* set U whose exterior measure is only ε larger than that of E. By monotonicity we also have $|E|_e \le |U|_e$, so the measure of U is very close to the measure of E.

Theorem 1.17. *Given $E \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, there exists an open set $U \supseteq E$ such that*

$$|E|_e \;\le\; |U|_e \;\le\; |E|_e + \varepsilon.$$

Consequently,

$$|E|_e \;=\; \inf\bigl\{|U|_e : U \text{ is open and } U \supseteq E\bigr\}. \tag{1.4}$$

Proof. If $|E|_e = \infty$, take $U = \mathbb{R}^d$. Otherwise $|E|_e < \infty$, so by definition of exterior measure there must exist boxes Q_k such that $E \subseteq \cup Q_k$ and $\sum \mathrm{vol}(Q_k) < |E|_e + \frac{\varepsilon}{2}$. Let Q_k^* be a larger box that contains Q_k in its interior and satisfies $\mathrm{vol}(Q_k^*) \le \mathrm{vol}(Q_k) + 2^{-k-1}\varepsilon$. Let U be the union of the interiors of the boxes Q_k^*. Then U is open, $E \subseteq U$, and

$$|U|_e \;\le\; \sum_k \mathrm{vol}(Q_k^*) \;\le\; \sum_k \mathrm{vol}(Q_k) \;+\; \frac{\varepsilon}{2} \;<\; |E|_e + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; |E|_e + \varepsilon. \qquad \square$$

If E has infinite exterior measure, then the set U constructed in Theorem 1.17 also has infinite measure. On the other hand, if $|E|_e < \infty$, then by applying Theorem 1.17 with $\frac{\varepsilon}{2}$ in place of ε we obtain the following corollary.

Corollary 1.18. *Suppose $E \subseteq \mathbb{R}^d$ satisfies $|E|_e < \infty$. Then for each $\varepsilon > 0$, there exists an open set $U \supseteq E$ such that*

$$|E|_e \;\le\; |U|_e \;<\; |E|_e + \varepsilon. \qquad \diamondsuit$$

Additional Problems

1.1. Show that $|\emptyset|_e = 0$ and $|\mathbb{R}^d|_e = \infty$.

1.2. Given a box Q in $\mathbb{R}^d$, show that $|Q^\circ|_e = |Q|_e$.

1.3. Show that if a set $E \subseteq \mathbb{R}^d$ has nonempty interior, then $|E|_e > 0$ (the converse does not hold in general, see Problem 1.25).

1.4. Let Z be a subset of $\mathbb{R}$ such that $|Z|_e = 0$. Set $Z^2 = \{x^2 : x \in Z\}$, and show that $|Z^2|_e = 0$.

1.5. Find the exterior measures of the following subsets of $\mathbb{R}^2$.

(a) $L = \{(x, x) : 0 \le x \le 1\}$, the diagonal of the unit square in $\mathbb{R}^2$.

(b) An arbitrary line segment, ray, or line in $\mathbb{R}^2$.

1.6. Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}^d$, and define

$$\limsup E_k \;=\; \bigcap_{j=1}^{\infty}\biggl(\bigcup_{k=j}^{\infty} E_k\biggr), \qquad \liminf E_k \;=\; \bigcup_{j=1}^{\infty}\biggl(\bigcap_{k=j}^{\infty} E_k\biggr).$$

(a) Show that $\limsup E_k$ consists of those points $x \in \mathbb{R}^d$ that belong to infinitely many E_k, while $\liminf E_k$ consists of those x which belong to all but finitely many E_k (i.e., there exists some $k_0 \in \mathbb{N}$ such that $x \in E_k$ for all $k \ge k_0$).

(b) Prove the *Borel–Cantelli Lemma*: If $\sum |E_k|_e < \infty$ then $\liminf E_k$ and $\limsup E_k$ have exterior Lebesgue measure zero.

(c) Given a set $E \subseteq \mathbb{R}^d$, show that $|E|_e = 0$ if and only if there exist countably many boxes Q_k such that each point $x \in E$ belongs to infinitely many Q_k and $\sum \mathrm{vol}(Q_k) < \infty$.

1.7. Let C be the Cantor set constructed in Example 1.8.

(a) The *ternary expansion* of $x \in [0,1]$ is

$$x = \sum_{n=1}^{\infty} \frac{c_n}{3^n},$$

where each c_n is either 0, 1, or 2. Every point $x \in [0,1]$ has a unique ternary expansion, except for points of the form $x = m/3^n$ with m, n integer, which have two ternary expansions. Show that x belongs to C if and only if x has at least one ternary expansion for which every c_n is either 0 or 2, and use this to show that C is uncountable.

(b) Show that C is perfect, C contains no interior points, and $C = \partial C$.

(c) Set

$$D = \left\{ \sum_{n=1}^{\infty} \frac{c_n}{3^n} : c_n = 0, 1 \right\}.$$

Show that $D + D = [0,1]$, and use this to show that $C + C = [0,2]$.

(d) Suppose $U \subseteq \mathbb{R}$ is an open, bounded set, and let $\overline{U}$ be its closure. Must $\overline{U} \backslash U$ be countable?

1.8. Modify the Cantor middle-thirds set construction as follows. Fix a parameter $0 < \theta < 1$, and at stage n remove intervals of relative length θ from F_n to form F_{n+1}. Show that the generalized Cantor set $C_\theta = \cap F_n$ is perfect, has no interior, equals its own boundary, and satisfies $|C_\theta|_e = 0$.

1.9. Show that a nonempty perfect subset of $\mathbb{R}^d$ must be uncountable.

1.10. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle in the complex plane. Fix $z \in S^1$, and define $T\colon S^1 \to S^1$ by $T(x) = zx$. Given $x \in S^1$, the set $\mathcal{O}(x) = \{z^n x\}_{n \ge 0}$ is called the *forward orbit* of x under T, and the *cluster set* of x under T is

$$\mathcal{A}(x) = \bigcap_{k \ge 0} \overline{\mathcal{O}(z^k x)} = \bigcap_{k \ge 0} \overline{\{z^n x\}_{n \ge k}}.$$

Prove the following statements.

(a) $\overline{\mathcal{O}(x)} = \mathcal{O}(x) \cup \mathcal{A}(x)$.

(b) T maps $\mathcal{A}(x)$ into itself.

For the remainder of this problem assume that $\mathcal{O}(x)$ is compact. Then from part (a) we obtain $\mathcal{A}(x) \subseteq \mathcal{O}(x)$, so there is a smallest nonnegative integer n_0 such that $z^{n_0} x \in \mathcal{A}(x)$. Prove the following statements.

(c) $\mathcal{O}(z^{n_0} x) = \mathcal{A}(x) = \mathcal{A}(z^{n_0} x)$.

(d) If $\mathcal{A}(x)$ is infinite then it is perfect (where we identify $\mathbb{C}$ with $\mathbb{R}^2$).

(e) $\mathcal{O}(x)$ is finite.