1.3 Equivalent Formulations of Lebesgue Measurability

The collection $\mathcal{L}_{\mathbb{R}^d}$ of Lebesgue measurable subsets of $\mathbb{R}^d$ is closed under both countable unions and complements. Since $\mathcal{L}_{\mathbb{R}^d}$ contains all of the open and closed subsets of $\mathbb{R}^d$, it also contains all of the following types of sets.

Definition 1.34. (a) A set $H \subseteq \mathbb{R}^d$ is a G_δ *set* if there exist countably many open sets U_k such that $H = \cap U_k$.

(b) A set $H \subseteq \mathbb{R}^d$ is an F_σ *set* if there exist countably many closed sets F_k such that $H = \cup F_k$. $\diamondsuit$

The symbol σ in this definition reminds us of the word "sums" and hence unions, while δ suggests the word "difference" and hence intersections.[1] Since Definition 1.34 only uses the concepts of countable unions and intersections of open and closed sets, it has a natural generalization to any topological space, i.e., we can define F_σ and G_δ sets in any space X that has a topology. In particular, every metric space has an associated topology. However, for the most part we will only be concerned with $X = \mathbb{R}^d$.

The half-open interval $[a, b)$ is neither an open nor a closed subset of $\mathbb{R}$, but it is a G_δ set because we can write

$$[a, b) \;=\; \bigcap_{k=1}^{\infty} \bigl(a - \tfrac{1}{k}, b\bigr).$$

It is also an F_σ set because

$$[a, b) \;=\; \bigcup_{k=1}^{\infty} \bigl[a, b - \tfrac{1}{k}\bigr].$$

Here are some additional examples of G_δ and F_σ sets.

Example 1.35. (a) Since the set of rationals is countable, let $\mathbb{Q} = \{r_n\}_{n\in\mathbb{N}}$ be an enumeration of all the rationals. Then $U_n = \mathbb{R}\backslash\{r_n\}$ is an open set that does not contain the rational r_n, and the intersection of these sets is $\cap U_n = \mathbb{R}\backslash\mathbb{Q}$, the set of irrationals. Hence the set of irrationals is a G_δ set.

(b) Could the set of rationals $\mathbb{Q}$ also be a G_δ set? If it was, then we could write $\mathbb{Q} = \cap V_n$ where each V_n is open. Each set V_n contains $\mathbb{Q}$, and therefore is dense in $\mathbb{R}$. The set U_n defined above is also dense in in $\mathbb{R}$, and

$$\biggl(\bigcap_{n=1}^{\infty} V_n\biggr) \cap \biggl(\bigcap_{n=1}^{\infty} U_n\biggr) \;=\; \mathbb{Q} \cap (\mathbb{R}\backslash\mathbb{Q}) \;=\; \emptyset.$$

[1] More precisely, F_σ is derived from the French words *fermé* (closed) and *somme* (union), while G_δ is derived from the German *Gebiet* (area, neighborhood, open set) and *Durchschnitt* (average, intersection).

However, since the real line is a complete metric space, the *Baire Category Theorem* [Hei11b, Thm. 3.96] implies that a countable intersection of open, dense subsets of $\mathbb{R}$ must be nonempty. This is a contradiction, so we conclude that $\mathbb{Q}$ cannot be a G_δ set. $\square$

We can keep going and define an $F_{\sigma\delta}$ set to be a countable intersection of F_σ sets, a $G_{\delta\sigma}$ set to be a countable union of G_δ sets, an $F_{\sigma\delta\sigma}$ set to be a countable union of $F_{\sigma\delta}$ sets, and so forth. All of these are Lebesgue measurable, though the collection of all of these sets does not exhaust the family $\mathcal{L}_{\mathbb{R}^d}$. Not only are all the sets in the various G_δ and F_σ families measurable, but they all belong to a special class of sets known as the *Borel σ-algebra*. We will explore Borel sets in more detail in Chapter 2 (see Definition 2.4).

Our next lemma shows that every set E, measurable or not, can be surrounded by a G_δ set that has exactly the same measure as E.

Lemma 1.36. *Given a set $E \subseteq \mathbb{R}^d$, there exists a G_δ set $H \supseteq E$ such that $|E|_e = |H|$.*

Proof. For each $k \in \mathbb{N}$ there exists an open set $U_k \supseteq E$ such that

$$|U_k| \;<\; |E|_e + \frac{1}{k}.$$

Let $H = \cap U_k$. Then H is a G_δ set, and $E \subseteq H \subseteq U_k$ for every k. By monotonicity we therefore have $|E|_e \le |H| \le |U_k| \le |E|_e + \frac{1}{k}$. As this is true for every k, we conclude that $|E|_e = |H|$. $\square$

We *cannot* conclude from Lemma 1.36 that $|H\backslash E|_e = 0$! Indeed, the next result shows that this is an equivalent way of distinguishing Lebesgue measurable sets from nonmeasurable sets.

Theorem 1.37. *Given $E \subseteq \mathbb{R}^d$, the following statements are equivalent.*

(a) *E is Lebesgue measurable.*

(b) *For every $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $|E\backslash F|_e \le \varepsilon$.*

(c) *$E = H\backslash Z$ where H is a G_δ set and $|Z| = 0$.*

(d) *$E = H \cup Z$ where H is an F_σ set and $|Z| = 0$.*

Proof. (a) $\Rightarrow$ (c). Suppose that E is measurable. Then for each $k \in \mathbb{N}$ we can find an open set $U_k \supseteq E$ such that $|U_k\backslash E| < \frac{1}{k}$ (and not just $|U_k| < |E|_e + \frac{1}{k}$ as in the proof of Lemma 1.36). Let $H = \cap U_k$. Then H is a G_δ set, $H \supseteq E$, and $Z = H\backslash E \subseteq U_k\backslash E$ for every k. Hence $|Z|_e \le |U_k\backslash E| < \frac{1}{k}$ for every k, so $|Z| = 0$.

Exercise: Complete the proof of the remaining implications. $\square$

Thus, every Lebesgue measurable set can be obtained by taking a G_δ or F_σ set and adding or subtracting a set with measure zero. Some other equivalent formulations of measurability are given in the Additional Problems for this section, and in Section 1.5 we will derive another equivalent formulation known as *Carathéodory's Criterion*.

The next exercise applies Theorem 1.37 to prove the seemingly "obvious" fact that the measure of a Cartesian product $|E \times F|$ is the product of the measures of E and F. One *inequality* relating $|E \times F|$ to $|E|\,|F|$ is easy, for if $\{Q_k\}_k$ is a covering of E by boxes and $\{R_\ell\}_\ell$ is a covering of F by boxes then $\{Q_k \times R_\ell\}_{k,\ell}$ is a covering of $E \times F$ by boxes, so

$$|E \times F| \;\le\; \sum_{k,\ell} \operatorname{vol}(Q_k \times R_\ell) \;=\; \biggl(\sum_k \operatorname{vol}(Q_k)\biggr) \biggl(\sum_\ell \operatorname{vol}(R_\ell)\biggr).$$

Taking the infimum over all such coverings of E and F yields the inequality $|E \times F| \le |E|\,|F|$. However, it is not so easy to prove that equality holds. The next exercise proceeds through cases to show that this is the case (recall that we interpret $0 \cdot \infty$ and $\infty \cdot 0$ as 0).

Exercise 1.38. (a) Show that if $U \subseteq \mathbb{R}^m$ and $V \subseteq R^n$ are open sets, then $U \times V$ is an open subset of $\mathbb{R}^{m+n}$ and $|U \times V| = |U|\,|V|$.

(b) Show that if $G \subseteq \mathbb{R}^m$ and $H \subseteq R^n$ are bounded G_δ sets, then $G \times H$ is a G_δ set and $|G \times H| = |G|\,|H|$.

(c) Show that E is a bounded measurable subset of $\mathbb{R}^m$ and Z is a subset of $\mathbb{R}^n$ with measure zero, then $E \times Z$ is measurable and $|E \times Z| = 0 = |E|\,|Z|$.

(d) Show that $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ are bounded measurable sets, then $E \times F$ is measurable and $|E \times F| = |E|\,|F|$.

(e) Show that $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ are any measurable sets, then $E \times F$ is measurable and $|E \times F| = |E|\,|F|$. ♢

Additional Problems

1.13. Show that the complement of a G_δ set is an F_σ set, and the complement of an F_σ set is a G_δ set.

1.14. (a) Show that every countable set is an F_σ set.

(b) Is any countable set a G_δ set? Is every countable set a G_δ set? Is $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ a G_δ set?

1.15. Exhibit a subset of $\mathbb{R}^d$ that belongs to one of the classes $G_{\delta\sigma}$, $F_{\sigma\delta}$, $G_{\delta\sigma\delta}$ $F_{\sigma\delta\sigma}$, etc., but is not a G_δ or an F_σ set.

1.16. Given $E \subseteq \mathbb{R}^d$, define $d_E(x) = \operatorname{dist}(x, E)$ for $x \in \mathbb{R}^d$. Prove the following statements.

(a) d_E is continuous on $\mathbb{R}^d$.

(b) The set $E_r = \{x \in \mathbb{R}^d : \operatorname{dist}(x, E) < r\}$ is open for each $r > 0$.

(c) If $E \subseteq \mathbb{R}^d$ is closed, then $d_E(x) = 0$ if and only if $x \in E$.

(d) Every closed set in $\mathbb{R}^d$ is a G_δ set.

(e) Every open set in $\mathbb{R}^d$ is an F_δ set.

1.17. Given a function $f\colon \mathbb{R}^d \to \mathbb{R}$, the *oscillation of f at x* is defined to be

$$\operatorname{osc}_f(x) \;=\; \inf_{\delta>0} \sup\bigl\{|f(y) - f(z)| : y, z \in B_\delta(x)\bigr\},$$

where $B_\delta(x)$ is the open ball with radius δ centered at x. Prove the following statements.

(a) f is continuous at x if and only if $\operatorname{osc}_f(x) = 0$.

(b) The set $\{x \in \mathbb{R}^d : \operatorname{osc}_f(x) \geq \varepsilon\}$ is closed for each $\varepsilon > 0$.

(c) $D = \{x \in \mathbb{R}^d : f \text{ is discontinuous at } x\}$ is an F_σ set, and therefore the set of continuities of f is a G_δ set.

1.18. (a) Does there exist a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at each rational point and discontinuous at each irrational?

(b) Does there exist a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at each irrational point and discontinuous at each rational?

1.19. Define the *inner Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ to be

$$|E|_i \;=\; \sup\bigl\{|F| : F \text{ closed, } F \subseteq E\bigr\}.$$

(a) Show that if $|E|_e < \infty$, then E is Lebesgue measurable if and only if $|E|_e = |E|_i$.

(b) Assuming that nonmeasurable subsets of $\mathbb{R}^d$ exist (see Problem 1.29), show that part (a) can fail if $|E|_e = \infty$.

1.20. Given a set $E \subseteq \mathbb{R}^d$ with $|E|_e < \infty$, show that the following statements are equivalent.

(a) E is Lebesgue measurable.

(b) For each $\varepsilon > 0$ we can write $E \;=\; (S \cup N_1) \setminus N_2$ where S is a union of finitely many nonoverlapping boxes and $|N_1|_e,\, |N_2|_e < \varepsilon$.

(c) For each $\varepsilon > 0$ there exists a set S that is a finite union of boxes and satisfies $|E \triangle S|_e < \varepsilon$.