1.4 Properties of Lebesgue Measure

Now we will prove that Lebesgue measure is a measure on $\mathbb{R}^d$ (in the sense of Definition 1.1) with respect to the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$.

Theorem 1.39 (Countable Additivity of Lebesgue Measure). *Suppose that $E_1, E_2, \dots$ are disjoint Lebesgue measurable subsets of $\mathbb{R}^d$. Then*

$$\left|\bigcup_{k=1}^{\infty} E_k\right| \;=\; \sum_{k=1}^{\infty} |E_k|.$$

Proof. Step 1. Assume that each set E_k is bounded. By subadditivity,

$$\left|\bigcup_{k=1}^{\infty} E_k\right| \;\le\; \sum_{k=1}^{\infty} |E_k|,$$

so our goal is to prove the opposite inequality.

Fix $\varepsilon > 0$. Then, since E_k is measurable, Theorem 1.37 implies that there exists a closed set $F_k \subseteq E_k$ such that

$$|E_k \backslash F_k| \;<\; \frac{\varepsilon}{2^k}.$$

Since we have assumed that each E_k is bounded, the sets F_k are disjoint and compact. Therefore, by Exercise 1.28(b) combined with monotonicity, for each integer $N \in \mathbb{N}$ we have

$$\sum_{k=1}^{N} |F_k| \;=\; \left|\bigcup_{k=1}^{N} F_k\right| \;\le\; \left|\bigcup_{k=1}^{N} E_k\right| \;\le\; \left|\bigcup_{k=1}^{\infty} E_k\right|.$$

Taking the limit as $N \to \infty$, we obtain

$$\sum_{k=1}^{\infty} |F_k| \;=\; \lim_{N\to\infty} \sum_{k=1}^{N} |F_k| \;\le\; \left|\bigcup_{k=1}^{\infty} E_k\right|.$$

Finally,

$$\begin{aligned}
\sum_{k=1}^{\infty} |E_k| \;&=\; \sum_{k=1}^{\infty} |F_k \cup (E_k \backslash F_k)| \\
&\le\; \sum_{k=1}^{\infty} \bigl(|F_k| + |E_k \backslash F_k|\bigr) \\
&\le\; \sum_{k=1}^{\infty} \left(|F_k| + \frac{\varepsilon}{2^k}\right)
\end{aligned}$$

$$= \sum_{k=1}^{\infty} |F_k| \;+\; \varepsilon$$

$$\leq \left|\bigcup_{k=1}^{\infty} E_k\right| \;+\; \varepsilon,$$

and since ε is arbitrary this shows that

$$\sum_{k=1}^{\infty} |E_k| \;\leq\; \left|\bigcup_{k=1}^{\infty} E_k\right|.$$

Step 2. Now let the E_k be arbitrary disjoint measurable subsets of $\mathbb{R}^d$. For each $j,\, k \in \mathbb{N}$, set

$$E_k^j \;=\; \{x \in E_k : j-1 \leq |x| < j\}.$$

Then $\{E_k^j\}_{k,j}$ is a countable collection of disjoint measurable sets, and for each fixed $k \in \mathbb{N}$ we have $\cup_j E_k^j = E_k$. Since each set E_k^j is bounded, by applying Step 1 twice we obtain

$$\left|\bigcup_{k=1}^{\infty} E_k\right| \;=\; \left|\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} E_k^j\right| \;=\; \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |E_k^j| \;=\; \sum_{k=1}^{\infty}\left|\bigcup_{j=1}^{\infty} E_k^j\right| \;=\; \sum_{k=1}^{\infty} |E_k|. \quad \square$$

Corollary 1.40. *Lebesgue measure* $|\cdot|$ *is a measure on* $\mathbb{R}^d$ *with respect to the* σ*-algebra* $\mathcal{L}_{\mathbb{R}^d}$ *(in the sense of Definition 1.1).* $\diamondsuit$

Now we will derive some of the important properties of Lebesgue measure. We begin by improving on what we know about monotonicity, which tells us that if A is a measurable set that is contained in another measurable set B, then $|A| \leq |B|$. However, we can write B is the disjoint union of A and $B\backslash A$, and the set $B\backslash A = B \cap A^{\mathrm{C}}$ is measurable, so by additivity we know a little more, namely that

$$|B| \;=\; |A| + |B\backslash A|.$$

This equation is true whenever A and B are measurable sets such that $A \subseteq B$. We are tempted to conclude that $|B\backslash A| = |B| - |A|$, but of course this has no meaning if $|A|$ and $|B|$ are both infinite. On the other hand, the next lemma shows that this equality does hold when $|A|$ is finite (even if $|B|$ is infinite).

Lemma 1.41. *If* $A \subseteq B$ *are Lebesgue measurable sets and* $|A| < \infty$ *then*

$$|B\backslash A| = |B| - |A|,$$

in the sense that if $|B| < \infty$ *then both sides above are finite and equal, while if* $|B| = \infty$ *then both sides are* ∞. $\diamondsuit$

Now we consider a nested increasing sequence of sets $E_1 \subseteq E_2 \subseteq \cdots$. Is it true that the measures of the sets E_k will increase to the measure of $\cup E_k$?

Theorem 1.42 (Continuity from Below). *If $E_k \subseteq \mathbb{R}^d$ are measurable and $E_1 \subseteq E_2 \subseteq \cdots$, then*

$$\left| \bigcup_{k=1}^{\infty} E_k \right| = \lim_{k\to\infty} |E_k|.$$

Proof. If we set $E_0 = \emptyset$, then

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{j=1}^{\infty} (E_j \backslash E_{j-1}),$$

and the sets on the right-hand side above are measurable and disjoint. Therefore, by countable additivity,

$$\begin{aligned}
\left| \bigcup_{k=1}^{\infty} E_k \right| &= \left| \bigcup_{j=1}^{\infty} (E_j \backslash E_{j-1}) \right| \\
&= \sum_{j=1}^{\infty} |E_j \backslash E_{j-1}| \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} |E_j \backslash E_{j-1}| \\
&= \lim_{N\to\infty} \left| \bigcup_{j=1}^{N} (E_j \backslash E_{j-1}) \right| \\
&= \lim_{N\to\infty} |E_N|.
\end{aligned}$$

Here is an alternative proof. If $|E_k| = \infty$ for some k then there is nothing to prove, so we may assume that $|E_k|$ is finite for every k. In this case, we can turn the computation above into a telescoping sum:

$$\begin{aligned}
\left| \bigcup_{k=1}^{\infty} E_k \right| &= \left| \bigcup_{j=1}^{\infty} (E_j \backslash E_{j-1}) \right| \\
&= \sum_{j=1}^{\infty} |E_j \backslash E_{j-1}| \\
&= \sum_{j=1}^{\infty} \Bigl(|E_j| - |E_{j-1}| \Bigr) \\
&= \Bigl(\lim_{N\to\infty} |E_N| \Bigr) - |E_0| \\
&= \lim_{N\to\infty} |E_N|. \qquad \square
\end{aligned}$$

However, the analogue of Theorem 1.42 for nested decreasing sets does not always hold!

Example 1.43. Let $B_n(0) = \{x \in \mathbb{R}^d : |x| < n\}$ be the open ball in $\mathbb{R}^d$ centered at the origin with radius n, and set $E_n = B_n(0)^C = \mathbb{R}^d \backslash B_n(0)$. Then $E_1 \supseteq E_2 \supseteq \cdots$ and $E = \cap E_n = \emptyset$, so

$$\left|\bigcap_{k=1}^{\infty} E_k\right| = 0 \qquad \text{yet} \qquad \lim_{k\to\infty} |E_k| = \infty. \qquad \diamondsuit$$

The problem in this example is that nested sets having infinite measure can decrease to a set that has finite measure. The next exercise shows that "continuity from above" holds as long as the sets in the sequence have finite measure from some point onward.

Exercise 1.44 (Continuity from Above). Let E_k be measurable subsets of $\mathbb{R}^d$ for $k \in \mathbb{N}$. Show that if $E_1 \supseteq E_2 \supseteq \cdots$ and $|E_k| < \infty$ for some k, then

$$\left|\bigcap_{k=1}^{\infty} E_k\right| = \lim_{k\to\infty} |E_k|. \qquad \diamondsuit$$

Additional Problems

1.21. Show that if $\{Q_k\}$ is a countable collection of nonoverlapping boxes, then $\left|\cup Q_k\right| = \sum |Q_k|$.

1.22. Show that if A and B are any measurable subsets of $\mathbb{R}^d$, then

$$|A \cup B| + |A \cap B| = |A| + |B|.$$

1.23. Find the Lebesgue measures of the following subsets of $\mathbb{R}^2$.

(a) $A = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$.

(b) $B = \{(x,y) : 0 \le x \le 1, 0 \le y \le x^2\}$.

1.24. Show that the Lebesgue measure of an arbitrary rectangle or parallelogram in $\mathbb{R}^2$ coincides with its volume in the usual sense.

1.25. Modify the Cantor middle-thirds set construction by removing stage n intervals of relative length θ_n from F_n to form F_{n+1}. Show that the generalized Cantor set $P = \cap F_n$ is perfect and has measure $|P| = 1 - \delta$ where $\delta = \prod_{k=1}^{\infty}(1-\theta_k)$. Conclude that if $\theta_k \to 0$ quickly enough, then P has *positive measure*.

1.26. Show that continuity from below holds for Lebesgue *exterior* measure, i.e., if $E_1 \subseteq E_2 \subseteq \cdots$ is *any* nested increasing sequence of subsets of $\mathbb{R}^d$, then $|\cup E_k|_e = \lim_{k\to\infty} |E_k|_e$ (in contrast, Problem 1.31 shows that continuity from above can fail for exterior measure, even if $|E_k|_e < \infty$ for every k).