1.5 Carathéodory's Criterion for Lebesgue Measurability

Although we have finished constructing Lebesgue measure on $\mathbb{R}^d$, there is still an important observation to make that will help us when we consider abstract outer measures in Chapter 2.

Take another look at Theorem 1.37, which gives several equivalent characterizations of Lebesgue measurability. Each of these characterizations is "topological" in some way, since they are formulated in terms of open, closed, G_δ, or F_σ sets. In contrast, the statement of *Carathéodory Criterion* for measurability only involves the definition of exterior Lebesgue measure. This makes this seemingly esoteric characterization quite important.

Theorem 1.45 (Carathéodory's Criterion). *A set $E \subseteq \mathbb{R}^d$ is Lebesgue measurable if and only if*

$$\forall\, A \subseteq \mathbb{R}^d, \quad |A|_e \;=\; |A \cap E|_e + |A \backslash E|_e. \qquad \diamondsuit \tag{1.8}$$

That is, a set E is measurable if and only if it has the property that when *any* other set A is given, the exterior measures of the two disjoint pieces $A \cap E$ and $A \backslash E$ that E cuts A into must exactly add up to the exterior measure of A.

Proof (of Theorem 1.45). $\Rightarrow$. Suppose that E is measurable and A is any subset of $\mathbb{R}^d$. Since $A = (A \cap E) \cup (A \backslash E)$, we have from subadditivity that

$$|A|_e \;\le\; |A \cap E|_e + |A \backslash E|_e.$$

By Lemma 1.36, we can find a G_δ set $H \supseteq A$ such that $|H| = |A|_e$. Note that we can write H as the disjoint union $H = (H \cap E) \cup (H \backslash E)$. Since Lebesgue measure is countably additive and H, E are measurable, we therefore have

$$|A|_e \;=\; |H| \;=\; |H \cap E| + |H \backslash E| \;\ge\; |A \cap E|_e + |A \backslash E|_e,$$

where the final inequality follows from monotonicity.

$\Leftarrow$. Assume that E is a bounded set such that equation (1.8) holds. Let $H \supseteq E$ be a G_δ set such that $|H| = |E|_e$. Then equation (1.8) implies that

$$|E|_e \;=\; |H| \;=\; |H \cap E|_e + |H \backslash E|_e \;=\; |E|_e + |H \backslash E|_e.$$

Since $|E|_e < \infty$, we conclude that $Z = H \backslash E$ has zero exterior measure and hence is measurable. Since $E = H \backslash Z$, it is measurable as well.

Exercise: Finish the proof for arbitrary sets E by considering the sets $E_k = \{x \in E : |x| \le k\}$. $\square$

1.6 Almost Everywhere

With great regularity, we encounter phenomena where sets of measure zero "don't matter." We introduce a terminology to deal with this situation.

Notation 1.46. A property that holds except possibly on a set of measure zero is said to hold *almost everywhere*, abbreviated a.e. ◇

For example, if C is the classical Cantor middle-thirds set, then $|C| = 0$ (Problem 1.7), so the characteristic function χ_C satisfies $\chi_C(x) = 0$ except for those x that belong to the zero measure set C. Therefore we say that $\chi_C(x) = 0$ for almost every x, or $\chi_C = 0$ a.e. for short.

The next definition gives an example of a quantity that is defined in terms of a property that holds almost everywhere.

Definition 1.47 (Essential Supremum). The *essential supremum* of an extended real-valued function $f\colon E \to [-\infty, \infty]$ is

$$\operatorname*{ess\,sup}_{x\in E} f(x) \;=\; \inf\bigl\{M : f(x) \le M \text{ a.e.}\bigr\}. \tag{1.9}$$

We say that f is *essentially bounded* if

$$\operatorname*{ess\,sup}_{x\in E} |f(x)| \;<\; \infty. \qquad \diamondsuit$$

Additional Problems

1.27. Show that the infimum in equation (1.9) is achieved, i.e., if we set $M = \operatorname{ess\,sup}_{x\in E} f(x)$ then $f(x) \le M$ a.e. In particular,

$$\operatorname*{ess\,sup}_{x\in E} |f(x)| \;=\; 0 \qquad \Longleftrightarrow \qquad f = 0 \text{ a.e.}$$

1.28. Show that if $Z \subseteq \mathbb{R}^d$ satisfies $|Z| = 0$, then $\mathbb{R}^d \backslash Z$ is dense in $\mathbb{R}^d$.