1.7 A Nonmeasurable Set

We will prove in this section that there exist subsets of the real line that are not Lebesgue measurable. The same idea shows that nonmeasurable sets exist in $\mathbb{R}^d$ for any dimension d (Problem 1.29). The proof is nonconstructive, in the sense that we must appeal to the Axiom of Choice to prove the existence of a nonmeasurable set.

First, we need to prove that Lebesgue measurable sets have a property that may seem rather surprising at first glance. Although a measurable set $E \subseteq \mathbb{R}$ need not contain any intervals, we will show that if E is measurable and has positive measure, then the *set of differences* of points from E must contain an interval.

Theorem 1.48. *If $E \subseteq \mathbb{R}$ is Lebesgue measurable and $|E| > 0$, then*

$$E - E \;=\; \{x - y : x, y \in E\}$$

contains an interval around 0.

Proof. Since $|E| > 0$, given $0 < \varepsilon < 1$ there exists some open set $U \supseteq E$ such that $|U| < (1+\varepsilon)\,|E|$. By Problem 1.11, we can write U as a countable union of nonoverlapping closed intervals I_k, so we have

$$U \;=\; \bigcup_{k=1}^{\infty} I_k \qquad \text{and} \qquad |U| \;=\; \sum_{k=1}^{\infty} |I_k|.$$

Set $E_k = E \cap I_k$, and note that E_k is measurable since E and I_k are both measurable. Because the intervals I_k are nonoverlapping, it follows that

$$E \;=\; \bigcup_{k=1}^{\infty} E_k \qquad \text{and} \qquad |E| \;=\; \sum_{k=1}^{\infty} |E_k|.$$

But $|U| < (1+\varepsilon)\,|E|$, so there must be at least one integer k such that

$$|I_k| \;\le\; (1+\varepsilon)\,|E_k|. \tag{1.10}$$

For the rest of the proof, we work with one such value of k.

Since I_k is an interval, write it as $I_k = [a_k, b_k]$. If $t \ge 0$ then $I_k \cup (I_k + t) \subseteq [a_k, b_k + t]$, while if $t \le 0$ then $I_k \cup (E_k + t) \subseteq [a_k - t, b_k]$. In any case, we see that

$$|I_k \cup (I_k + t)| \;\le\; |I_k| + |t|.$$

Consequently, if E_k and $E_k + t$ are *disjoint* then we have

$$\begin{aligned} 2\,|I_k| &\le 2\,(1+\varepsilon)\,|E_k| \\ &= (1+\varepsilon)\,|E_k \cup (E_k+t)| \\ &\le (1+\varepsilon)\,|I_k \cup (I_k+t)| \\ &\le (1+\varepsilon)\,\big(|I_k| + |t|\big). \end{aligned}$$

This equation cannot hold when $|t|$ is small, so E_k and $E_k + t$ must intersect for all small enough $|t|$. Specifically,

$$|t| < \frac{1-\varepsilon}{1+\varepsilon}\,|I_k| \quad \Longrightarrow \quad E_k \cap (E_k+t) \neq \emptyset.$$

Hence $t \in E_k - E_k$ for all small enough $|t|$. Therefore $E_k - E_k$ contains an interval, so $E - E$ does as well. □

Now we will demonstrate the existence of a nonmeasurable set. Note that any such set must be uncountable, since every set with exterior measure zero is Lebesgue measurable.

Theorem 1.49. *There exists a nonmeasurable subset N of $\mathbb{R}$.*

Proof. Define an equivalence relation $\sim$ on $\mathbb{R}$ by declaring that

$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Q}. \tag{1.11}$$

As for any equivalence relation, the distinct equivalence classes partition the set $\mathbb{R}$. For the relation $\sim$, the equivalence class of a point $x \in \mathbb{R}$ is

$$[x] = \{y \in \mathbb{R} : x \sim y\} = \{x + r : r \in \mathbb{Q}\} = x + \mathbb{Q}.$$

The Axiom of Choice implies that there exists a set $N \subseteq \mathbb{R}$ that contains exactly one element of each distinct equivalence class of $\sim$. Since the distinct equivalence classes partition $\mathbb{R}$, we have

$$\mathbb{R} = \bigcup_{x\in N} [x] = \bigcup_{x\in N} (x+\mathbb{Q}) = \bigcup_{r\in\mathbb{Q}} (r+N). \tag{1.12}$$

Therefore, by subadditivity,

$$\infty = |\mathbb{R}|_e \le \sum_{r\in\mathbb{Q}} |N+r|_e = \sum_{r\in\mathbb{Q}} |N|_e.$$

In particular, we must have $|N|_e > 0$. However, any two distinct points x, y in N must come from distinct equivalence classes, and hence must differ by an irrational amount. Consequently, $N - N$ contains no intervals. Theorem 1.48 therefore implies that N is not Lebesgue measurable. □

There are several ways to obtain other nonmeasurable sets from N. For example, if we set

$$N_k = N \cap [k, k+1), \qquad k \in \mathbb{Z}, \tag{1.13}$$

then $N = \cup_{k\in\mathbb{Z}} N_k$. Hence at least one set N_k must be nonmeasurable. By translating, this gives us a nonmeasurable subset of the interval $[0,1]$. Problem 1.32 extends this further: Every set with positive exterior measure must contain a nonmeasurable subset.

We can use the existence of bounded nonmeasurable sets to show that exterior Lebesgue measure is not finitely additive.

Example 1.50. Let N be a bounded nonmeasurable set. Then, by definition of measurable set, there exists some $\varepsilon > 0$ such that no matter what open set $U \supseteq N$ that we choose, we will have

$$|U \backslash N|_e \ \geq \ \varepsilon.$$

However, since N has finite exterior measure, Corollary 1.18 implies that there is some open set $U \supseteq N$ such that

$$|N|_e \ \leq \ |U| \ < \ |N|_e + \varepsilon.$$

Therefore

$$U = N \cup (U \backslash N),$$

and the sets N and $U \backslash N$ are disjoint, yet we have

$$|N \cup (U\backslash N)|_e = |U|_e < |N|_e + \varepsilon \leq |N|_e + |U\backslash N|_e.$$

Therefore exterior measure is not finitely additive. ◇

Consider again the sets N_k defined in equation (1.13). Translating each set N_k back into the interval $[0,1]$ and taking the union gives us the set

$$\mathcal{N} = \bigcup_{k\in\mathbb{Z}} (N_k - k).$$

Just like N, the set $\mathcal{N}$ contains one representative of each distinct equivalence class of the relation $\sim$ defined by equation (1.11). However, unlike N, the set $\mathcal{N}$ is contained in the interval $[0,1)$. We can take this a step further by thinking of $[0,1)$ as a group under the operation of addition mod 1. Precisely, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and define

$$x \bmod 1 = x - \lfloor x \rfloor \qquad \text{(the fractional part of } x\text{)}.$$

Then $[0,1)$ is a group with respect to the operation $x \oplus y = (x+y) \bmod 1$. We can "translate" sets with respect to this operation to obtain new sets that are still contained within $[0,1)$. For example, given a rational number $r \in [0,1)$, set

$$\begin{aligned}\mathcal{N}_r &= \mathcal{N} \oplus r = \{(r+x) \bmod 1 : x \in \mathcal{N}\} \\ &= \Big((\mathcal{N}+r) \cap [0,1)\Big) \cup \Big((\mathcal{N}+r-1) \cap [0,1)\Big).\end{aligned}$$

Analogously to equation (1.12), we have

$$[0,1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} \mathcal{N}_r, \tag{1.14}$$

and the sets $\mathcal{N}_r$ in this union are disjoint. We will use this fact to prove the claim that we made in the very opening paragraphs of this chapter (though for simplicity of presentation, we restrict to dimension $d = 1$ only).

Theorem 1.51. *There does not exist a function* $\mu\colon \mathcal{P}(\mathbb{R}) \to [0,\infty]$ *such that:*

(a) μ *is countably additive,*

(b) μ *is translation-invariant, and*

(c) $\mu\big([0,1)\big) = 1$.

Proof. Suppose that there is such a function μ. Then the fact that μ is both translation-invariant and countably additive implies that $\mu(\mathcal{N}_r) = \mu(\mathcal{N})$ for every r. Using equation (1.14), we conclude that

$$1 = \mu\big([0,1)\big) = \sum_{r \in \mathbb{Q} \cap [0,1)} \mu(\mathcal{N}_r) = \sum_{r \in \mathbb{Q} \cap [0,1)} \mu(\mathcal{N}).$$

However, the rightmost series on the line above can only be 0 or ∞, so this is a contradiction. □

Additional Problems

1.29. Show that for each integer $d > 0$ there exists a set $N \subseteq \mathbb{R}^d$ that is not Lebesgue measurable.

1.30. Show that there exist disjoint sets $E_1, E_2, \dots$ in $\mathbb{R}$, each with positive exterior measure, such that

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e < \sum_{k=1}^{\infty} |E_k|_e,$$

with strict inequality.

1.31. Show that there exist sets $E_1 \supseteq E_2 \supseteq \cdots$ in $\mathbb{R}$ such that $|E_k|_e < \infty$ for every k and

$$\left|\bigcap_{k=1}^{\infty} E_k\right|_e < \lim_{k\to\infty} |E_k|_e,$$

with strict inequality. Hence continuity from above does not hold for exterior Lebesgue measure in general (compare Problem 1.26).

1.32. Show that every subset of $\mathbb{R}$ that has positive exterior Lebesgue measure contains a nonmeasurable subset.