1.8 Linear and Lipschitz Transformations

Our definition of measure has so far been tied to the orientation of the coordinate axes, in the sense that have take boxes with sides parallel to the coordinate axes as our starting point. It seems reasonable that Lebesgue measure should be unchanged if we use boxes with a different orientation. Equivalently, Lebesgue measure should be invariant under rotations, and for other linear transformation the change in measure should be related to the determinant of the matrix for the transformation. We prove these and some other facts in this section.

We must be careful not to take measurability for granted when we apply a transformation. In general, if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable and $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous, it need not be the case that the direct image $f(E) = \{f(x) : x \in E\}$ is a measurable subset of $\mathbb{R}^m$! We will give a specific counterexample in the following section (see Example 1.61). However, we show now that functions that possess a little "extra regularity" beyond just continuity do preserve measurability. We formulate this regularity criterion as follows (compare Problem 1.12, which introduced Lipschitz transformations for the case $m = n = 1$).

Definition 1.52 (Lipschitz Transformation). A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a *Lipschitz transformation* if there exists a constant $C > 0$ (called a *Lipschitz constant* for T) such that

$$|T(x) - T(y)| \;\le\; C\,|x - y|, \qquad x, y \in \mathbb{R}^n. \qquad \diamondsuit$$

Exercise 1.53. Prove the following statements.

(a) If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then it is uniformly continuous.

(b) Every linear function $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz.

(c) The translation map $\tau_a\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_a(x) = x - a$ is Lipschitz but not linear. $\diamondsuit$

Exercise 1.54. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz mapping. This exercise will prove that T preserves measurability of sets.

(a) Show that if $Z \subseteq \mathbb{R}^n$ satisfies $|Z| = 0$, then $|T(Z)| = 0$.

(b) Use the fact that T is continuous to prove that if $F \subseteq \mathbb{R}^n$ is compact, then $T(F) \subseteq \mathbb{R}^m$ is compact.

(c) Let H be an F_σ set in $\mathbb{R}^n$. Show that $T(H)$ is an F_σ set in $\mathbb{R}^m$.

(d) Show that if E is a measurable subset of $\mathbb{R}^n$, then $T(E)$ is a measurable subset of $\mathbb{R}^m$. $\diamondsuit$

If we specialize to a linear function $L\colon \mathbb{R}^n \to \mathbb{R}^m$ then we can give an explicit formula for the measure of $L(E)$ in terms of the measure of E. A linear mapping on Euclidean space is given by a matrix, i.e., there exists an $m \times n$ matrix A such that $L(x) = Ax$. We define the determinant of the mapping L to be the determinant of the matrix A.

Theorem 1.55. *If $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and $E \subseteq \mathbb{R}^n$ is Lebesgue measurable, then $L(E)$ is a measurable subset of $\mathbb{R}^m$ and*

$$|L(E)| \;=\; |\det(L)|\,|E|.$$

Proof. Since linear mappings are Lipschitz, Exercise 1.54 implies that $L(E)$ is a measurable subset of $\mathbb{R}^m$.

Suppose first that $|E| < \infty$. Given $\varepsilon > 0$, we can find at most countably many boxes Q_k such that $E \subseteq \cup Q_k$ and $\sum |Q_k| \le |E| + \varepsilon$. Each set $L(Q_k)$ is a parallelepiped in $\mathbb{R}^m$, and $|L(Q_k)| = |\det(L)|\,|Q_k|$ by Problem 1.33. Therefore

$$|L(E)| \;\le\; \sum_k |L(Q_k)| \;\le\; \sum_k |\det(L)|\,|Q_k| \;\le\; |\det(L)|\,\bigl(|E| + \varepsilon\bigr).$$

Since ε is arbitrary, this implies that

$$|L(E)| \;=\; |\det(L)|\,|E|. \tag{1.15}$$

If $\det(L) = 0$ then we are done. If $\det(L) \ne 0$ then L is invertible, so by applying equation (1.15) to the linear mapping L^{-1} and the measurable set $L(E)$, we see that

$$|E| \;=\; |L^{-1}(L(E))| \;\le\; |\det(L^{-1})|\,|L(E)| \;=\; \frac{|L(E)|}{|\det(L)|}.$$

Hence $|L(E)| = |\det(L)|\,|E|$ for all sets E that have finite measure.

Exercise: Extend to arbitrary sets E by writing E as a disjoint union of sets with finite measures. □

Corollary 1.56. *The Lebesgue measure of $E \subseteq \mathbb{R}^n$ is invariant under rotations.* ◊

Additional Problems

1.33. Show that if Q is a box in $\mathbb{R}^n$ and $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $|L(Q)| = |\det(L)|\,|Q|$.