1.9 The Cantor–Lebesgue Function

We will construct an important function in this section through an iterative procedure that is related to the construction of the Cantor set as given in Example 1.8.

Consider the two functions φ_1, φ_2 pictured in Figure 1.2. The function φ_1 takes the constant value $\frac{1}{2}$ on the interval $(\frac{1}{3}, \frac{2}{3})$ that is removed from $[0, 1]$ in the first stage of the construction of the Cantor middle-thirds set, and is linear on the remaining intervals. The function φ_2 takes the same constant $\frac{1}{2}$ on the interval $(\frac{1}{3}, \frac{2}{3})$ but additionally is constant with values $\frac{1}{4}$ and $\frac{3}{4}$ on the two intervals that are removed in the second stage of the construction of the Cantor set. We continue this process and define $\varphi_3, \varphi_4, \dots$ in a similar way. Each function φ_k is continuous, and is constant on each of the open intervals that were removed at the kth stage of the construction of the Cantor set. The following exercise shows that these functions converge uniformly to a continuous function.

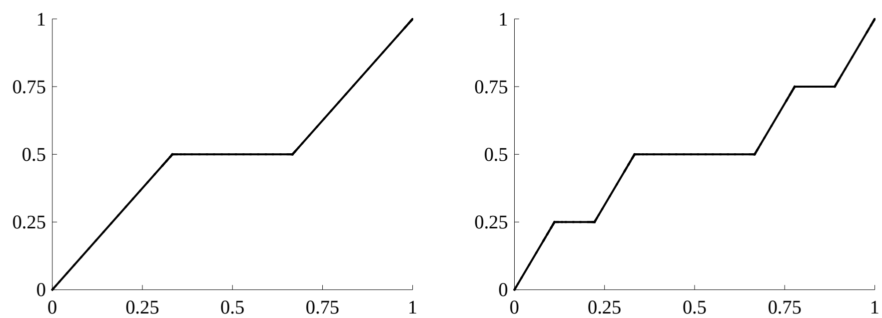

Fig. 1.2. Top left: The function φ_1. Top right: The function φ_2.

Exercise 1.57. Prove the following facts.

(a) Each function φ_k is monotone increasing on the interval $[0, 1]$, and $|\varphi_{k+1}(x) - \varphi_k(x)| < 2^{-k}$ for every $x \in [0, 1]$.

(b) The functions φ_k converge uniformly on $[0, 1]$, and the limit function $\varphi(x) = \lim_{k\to\infty} \varphi_k(x)$ is continuous on $[0, 1]$. Moreover, φ is differentiable at almost every point $x \in [0, 1]$, and although φ is not differentiable at all points, we have $\varphi'(x) = 0$ a.e. in $[0, 1]$. ◇

This limit function φ is called the *Cantor–Lebesgue function* or, more picturesquely, the *Devil's staircase*. If we extend φ to $\mathbb{R}$ by reflecting it about the point $x = 1$ and declaring it to be zero outside of $[0, 2]$, we obtain the continuous function φ that is pictured in Figure 1.3.

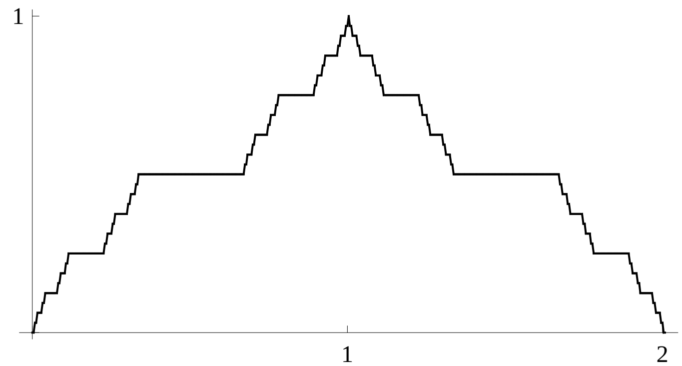

Fig. 1.3. The reflected Devil's staircase (Cantor–Lebesgue function).

The Cantor–Lebesgue function is not Lipschitz, but it does satisfy a weaker condition.

Definition 1.58. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *Hölder continuous* on $\mathbb{R}$ with exponent $\alpha > 0$ if there exists a constant $C > 0$ such that

$$\forall\, x, y \in \mathbb{R}, \quad |f(x) - f(y)| \;\le\; C\,|x-y|^{\alpha}. \qquad \diamondsuit$$

Thus Lipschitz continuity on $\mathbb{R}$ is Hölder continuity with exponent $\alpha = 1$.

Exercise 1.59. Prove that the Cantor–Lebesgue function is Hölder continuous on the interval $[0,1]$ precisely for exponents α that lie in the range $0 < \alpha \le \log_3 2 \approx 0.6309\ldots$. Conclude that while φ is continuous and monotone increasing on $[0,1]$, it is not Lipschitz. $\diamondsuit$

We will use the Cantor–Lebesgue function to derive some interesting insights into the behavior of measurable sets under continuous functions. First we show that a continuous function can map a set with zero measure to a set with positive measure.

Lemma 1.60. *The Cantor–Lebesgue function φ maps the Cantor set C, which has zero measure, to a set that has positive Lebesgue measure.*

Proof. If $x \notin C$, then x belongs to one of the open intervals removed at some stage in forming the Cantor set. Consequently $\varphi(x)$ is a dyadic rational number, i.e., $\varphi(x) = m/2^n$ for some integers m and n. Therefore φ maps the complement of the Cantor set into the set of rationals in $[0,1]$, which is a countable set. Consequently $\varphi(C)$ includes all of the irrational numbers in $[0,1]$, so $\varphi(C) = [0,1]\backslash Z$ where $Z \subseteq \mathbb{Q}$. Since Z has measure zero, it follows that $\varphi(C)$ is measurable and $|\varphi(C)| = 1$. $\square$

Second, we show that a continuous function need not map a measurable set to a measurable set.

Lemma 1.61. *Let φ be the Cantor–Lebesgue function. There exists a measurable set $E \subseteq [0,1]$ such that $\varphi(E)$ is not measurable.*

Proof. Let N be a nonmeasurable subset of $[0,1]$. By replacing N with $N \setminus \mathbb{Q}$, we may assume that N contains no rational numbers. Consequently $\varphi^{-1}(N)$ is contained in the Cantor set C. Since C has zero measure, monotonicity implies that $|\varphi^{-1}(N)| = 0$, so $E = \varphi^{-1}(N)$ is measurable. However, since φ is surjective, the image of E under φ is N, which is not measurable. □

Additional Problems

1.34. Show that if a function $f\colon \mathbb{R} \to \mathbb{R}$ is Hölder continuous for some exponent $\alpha > 1$, then f is constant.

1.35. Let C be the Cantor set and φ the Cantor–Lebesgue function. Define $g(x) = \varphi(x) + x$, and prove the following statements.

(a) Both $g\colon [0,1] \to [0,2]$ and $g^{-1}\colon [0,2] \to [0,1]$ are continuous, strictly increasing bijections.

(b) $g(C)$ is a closed subset of $[0,2]$, and $|g(C)| = 1$.

(c) Let N be a nonmeasurable subset of $g(C)$ (such a set exists by Problem 1.32). Then $A = g^{-1}(N)$ is Lebesgue measurable.

Remark: See Problem 2.10 for a further extension of this problem.