2.3 Basic Properties of Measures

This section presents some properties of abstract measures that are analogous to properties of Lebesgue measure that we encountered in Chapter 1. As most of the proofs of these properties are almost identical to those for Lebesgue measure, we state these results as exercises.

Exercise 2.16. Given a measure space (X, Σ, μ), prove the following statements.

(a) Monotonicity: If A, $B \in \Sigma$ and $A \subseteq B$ then $\mu(A) \le \mu(B)$.

(b) If A, $B \in \Sigma$, $A \subseteq B$, and $\mu(A) < \infty$ then $\mu(B \backslash A) = \mu(B) - \mu(A)$. $\diamondsuit$

Remark 2.17. In Chapter 6 we will study *signed measures*, which satisfy countable additivity but allow the measure to take values in the range $-\infty \le \mu(E) \le \infty$. An important difference between *measures* and *signed measures* is that monotonicity need not hold for signed measures! If μ is a signed measure and $A \subseteq B$ then, since $\mu(A)$ might be negative, we cannot infer from $\mu(A) + \mu(B \backslash A) = \mu(B)$ that $\mu(A) \le \mu(B)$. $\diamondsuit$

Since *measures* are monotonic, we immediately obtain the following corollary of Exercise 2.16.

Corollary 2.18. *Let (X, Σ, μ) be a measure space, and suppose that $E \in \Sigma$ satisfies $\mu(E) = 0$. If $A \subseteq E$ and $A \in \Sigma$, then $\mu(A) = 0$.* $\square$

Thus, all *measurable* subsets of a zero measure set have zero measure. In general, however, a set with zero measure may contain subsets that are not measurable. We give the following special name to measures that have the property that *every* subset of a measurable set with zero measure are measurable.

Definition 2.19 (Complete Measure). Let (X, Σ, μ) be a measure space. we say that μ is *complete* if

$$E \in \Sigma,\ \mu(E) = 0 \quad \Longrightarrow \quad A \in \Sigma \text{ for all } A \subseteq E. \qquad \diamondsuit$$

The reader should be aware that the terms "complete" and "algebra" are heavily overused in mathematics and appear in many unrelated definitions and contexts.

Example 2.20. If Z is a Lebesgue measurable subset of $\mathbb{R}^d$ that has measure zero, then every subset of Z is Lebesgue measurable (Lemma 1.21). Thus Lebesgue measure is complete, although we should emphasize that we are (as usual) implicitly taking the σ-algebra to be $\mathcal{L}_{\mathbb{R}^d}$.

If we change the σ-algebra, then Lebesgue measure restricted to this new σ-algebra may not be complete. For example, consider Lebesgue measure restricted to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$. If $Z \in \mathcal{B}_{\mathbb{R}^d}$ is a Borel set and $|Z| = 0$, then it is possible that Z may contain a subset A that is not a Borel set. This set A is not measurable *with respect to* $\mathcal{B}_{\mathbb{R}^d}$, so Lebesgue measure is not a complete measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. ♢

We will consider completeness of measures in more detail in Section 2.4. For now, we give a name to those sets E that have zero measure.

Notation 2.21 (Null Sets). Given a measure space (X, Σ, μ), a set $E \in \Sigma$ such that $\mu(E) = 0$ is called a *μ-null set* or a set of *μ-measure zero.*

A property that holds for all $x \in X$ except possibly for x in a μ-null set E is said to hold *μ-almost everywhere* (abbreviated μ-a.e.). ♢

We often omit writing the symbol μ if it is clear from context, e.g., we may simply say that E is a *null set* instead writing *μ-null set*, or say that a property holds *almost everywhere* instead of *μ-almost everywhere.*

Remark 2.22. Null sets can be "very large" in senses other than their measure. For example, for the δ-measure on $\mathbb{R}^d$ we have $\delta(\mathbb{R}^d \backslash \{0\}) = 0$. Thus $\mathbb{R}^d \backslash \{0\}$ is a δ-null set, and consequently $\chi_{\mathbb{R}^d \backslash \{0\}} = 0$ δ-a.e.

Abstract measures satisfy continuity from above and below in the following sense.

Exercise 2.23. Let (X, Σ, μ) be a measure space. Given measurable sets $E_1, E_2, \dots$ in X, prove that the following statements hold.

(a) Continuity from below: If $E_1 \subseteq E_2 \subseteq \cdots$, then

$$\mu\biggl(\bigcup_{k=1}^{\infty} E_k\biggr) \;=\; \lim_{k\to\infty} \mu(E_k).$$

(b) Continuity from above: If $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_k) < \infty$ for some k, then

$$\mu\biggl(\bigcap_{k=1}^{\infty} E_k\biggr) \;=\; \lim_{k\to\infty} \mu(E_k). \qquad \diamondsuit$$

The final property that we will discuss in this section is countable subadditivity. Here we require a different approach that what was used for Lebesgue measure. When we constructed Lebesgue measure, we first constructed exterior Lebesgue measure, which is subadditive, and then restricted to the Lebesgue measurable sets to obtain Lebesgue measure. Hence Lebesgue measure simply inherits subadditivity from exterior Lebesgue measure, and the difficulty with Lebesgue measure is proving that countable additivity holds on the Lebesgue measurable sets. In contrast, an abstract measure μ is countably additive by definition, and we must *deduce* countable subadditivity from that hypothesis.

Theorem 2.24 (Countable Subadditivity). *Let (X, Σ, μ) be a measure space. If $E_1, E_2, \ldots \in \Sigma$, then*

$$\mu\Bigl(\bigcup_k E_k\Bigr) \;\le\; \sum_k \mu(E_k).$$

Proof. Using the disjointization trick (Exercise 2.7), we can write $\cup E_k = \cup F_k$ where the sets F_k are measurable and disjoint, and $F_k \subseteq E_k$ for each k. Combining countable additivity and monotonicity, it follows that

$$\mu\Bigl(\bigcup_k E_k\Bigr) \;=\; \mu\Bigl(\bigcup_k F_k\Bigr) \;=\; \sum_k \mu(F_k) \;\le\; \sum_k \mu(E_k). \quad \square$$

Additional Problems

2.16. Show that every σ-finite measure is semifinite.

2.17. Suppose that (X, Σ, μ) is a measure space and $\{x\} \in \Sigma$ for every $x \in X$. Show that if μ is a finite measure, then $E = \{x \in X : \mu\{x\} > 0\}$ is countable.

2.18. Let (X, Σ, μ) be a measure space and fix $E_k \in \Sigma$ for $k \in \mathbb{N}$. Define

$$\limsup E_k \;=\; \bigcap_{j=1}^{\infty}\Bigl(\bigcup_{k=j}^{\infty} E_k\Bigr), \qquad \liminf E_k \;=\; \bigcup_{j=1}^{\infty}\Bigl(\bigcap_{k=j}^{\infty} E_k\Bigr).$$

Show that

$$\mu\Bigl(\liminf E_k\Bigr) \;\le\; \liminf \mu(E_k).$$

Also show that if $\mu(\cup E_k) < \infty$, then

$$\mu\Bigl(\limsup E_k\Bigr) \;\ge\; \limsup \mu(E_k).$$

2.19. Let (X, Σ, μ) be a measure space. Show that if μ is semifinite, $E \in \Sigma$, and $\mu(E) = \infty$, then for any $C > 0$ there exists some measurable set $A \subseteq E$ that satisfies $C < \mu(A) < \infty$.

2.20. Suppose that μ is a *finitely* additive function on a σ-algebra Σ of subsets of X, i.e., $\mu(\emptyset) = 0$ and $\mu\bigl(\cup_{k=1}^n E_k\bigr) = \sum_{k=1}^n \mu(E_k)$ for any finite collection of disjoint sets $E_1, \ldots, E_n \in \Sigma$. Prove the following statements.

(a) μ is a measure if and only if it satisfies continuity from below.

(b) If $\mu(X) < \infty$, then μ is a measure if and only if it satisfies continuity from above.