2.6 The Construction of an Outer Measure

Now we want to show now that if we are given a particular class of "elementary sets" $\mathcal{E}$ whose measures are specified, then we can extend this to an outer measure on X. We do this by employing the same technique that we used to create exterior Lebesgue measure, i.e., we cover arbitrary sets by countable unions of elementary sets in all possible ways.

Theorem 2.33. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a fixed collection of sets such that*

(a) $\emptyset \in \mathcal{E}$, *and*

(b) *there exist countably many sets $E_k \in \mathcal{E}$ such that $\cup E_k = X$.*

Suppose that $\rho\colon \mathcal{E} \to [0, \infty]$ satisfies $\rho(\emptyset) = 0$. For each $A \subseteq X$, define

$$\mu^*(A) \;=\; \inf\Big\{\sum_k \rho(E_k)\Big\}, \tag{2.7}$$

where the infimum is taken over all finite or countable covers of A by sets $E_k \in \mathcal{E}$. Then μ^* is an outer measure on X.

Proof. The given hypotheses ensure that every subset of X has at least one covering by elements of $\mathcal{E}$. Hence the infimum in equation (2.7) is not taken over the empty set, and therefore defines a value in $[0, \infty]$ for each $A \subseteq X$.

Since $\{\emptyset\}$ is one covering of $\emptyset$ by elements of $\mathcal{E}$, we have

$$0 \;\le\; \mu^*(\emptyset) \;\le\; \rho(\emptyset) \;=\; 0,$$

and monotonicity follows from the fact that if $A \subseteq B$ then every covering of B by sets $E_k \in \mathcal{E}$ is also a covering of A. Finally, the proof that μ^* is countably subadditive is just like the proof that exterior Lebesgue measure is countably subadditive, so we assign this as an exercise. □

We refer to the elements of the collection $\mathcal{E}$ in Theorem 2.33 as *elementary sets*. By Theorem 2.32, since the function μ^* constructed in Theorem 2.33 is an outer measure, we know that there is an associated σ-algebra Σ of μ^*-measurable sets, and we also know that $\mu = \mu^*|_\Sigma$ is a complete measure on (X, Σ). However, there are still two important questions that we have not addressed.

- Are the elementary sets measurable, i.e., do we have $\mathcal{E} \subseteq \Sigma$?
- Is $\mu^*(E) = \rho(E)$ for $E \in \mathcal{E}$?

Unfortunately, the following example shows that the answers to these questions are *no* in general.

Example 2.34. Let X be a set that contains at least two elements. Let E be a nonempty proper subset of E, and define $\mathcal{E} = \{\emptyset, E, E^{\mathrm{C}}, X\}$.

(a) If we define

$$\rho(\emptyset) = 0, \qquad \rho(E) = \frac{1}{4}, \qquad \rho(E^{\mathrm{C}}) = \frac{1}{4}, \qquad \rho(X) = 1,$$

then $\mu^*(X) = \frac{1}{2} \neq \rho(X)$, so the outer measure μ^* does not agree with ρ on the elementary sets.

(b) If we define

$$\rho(\emptyset) = 0, \qquad \rho(E) = 1 \qquad \rho(E^{\mathrm{C}}) = 1 \qquad \rho(X) = 1,$$

then

$$\mu^*(X) \;=\; 1 \;\neq\; 2 \;=\; \mu^*(X \cap E) + \mu^*(X \cap E^{\mathrm{C}}),$$

so X is not μ^*-measurable, even though it is an elementary set. ♢

Thus, we need to impose some extra conditions on the function ρ and the class $\mathcal{E}$ of elementary sets.

Definition 2.35 (Premeasure). Given a set X, let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an *algebra*, i.e., $\mathcal{A}$ is nonempty and is closed under complements and *finite* unions. A *premeasure* on $\mathcal{A}$ is a function $\mu_0 \colon \mathcal{A} \to [0,\infty]$ satisfying

(a) $\mu_0(\emptyset) = 0$, and

(b) if $E_1, E_2, \dots \in \mathcal{A}$ are disjoint and $\bigcup_{k=1}^\infty E_k \in \mathcal{A}$, then

$$\mu_0\biggl(\bigcup_{k=1}^\infty E_k\biggr) \;=\; \sum_{k=1}^\infty \mu_0(E_k). \qquad ♢$$

Note that in requirement (b) we are not assuming that $\mathcal{A}$ is closed under countable unions. We only require that *if* the union of the disjoint sets A_k belongs to $\mathcal{A}$ *then* μ_0 will be countably additive on those sets. On the other hand, since $\emptyset \in \mathcal{A}$, it is true that a premeasure is finitely additive. Consequently, if $A \subseteq B$ then $\mu_0(B) = \mu_0(A) + \mu_0(B \backslash A)$, and it follows from this that μ_0 is monotonic. We formalize this statement as a lemma.

Lemma 2.36. *A premeasure μ_0 on an algebra $\mathcal{A}$ is monotonic and finitely additive on $\mathcal{A}$.* ♢

Note that the collection of all boxes in $\mathbb{R}^d$ does not form an algebra, since it is not closed under either complements or finite unions. Thus it is not *entirely* obvious how the construction of Lebesgue measure from Chapter 1 relates to premeasures. We will consider this issue in detail in Section 6.1.

Given a premeasure μ_0, we associate the outer measure μ^* defined by

$$\mu^*(E) \;=\; \inf\biggl\{\sum_{k=1}^\infty \mu_0(E_k) : E_k \in \mathcal{A},\, E \subseteq \bigcup_k E_k\biggr\},$$

and we let Σ denote the corresponding σ-algebra of μ^*-measurable sets. Carathéodory's Theorem implies that $\mu = \mu^*|_\Sigma$ is a complete measure. Our next goal is to show that such a measure is "well-behaved."

Theorem 2.37. *Given a premeasure μ_0 on an algebra $\mathcal{A}$, let μ^* be the associated outer measure. Then the following statements hold.*

(a) $\mu^*|_{\mathcal{A}} = \mu_0$, *i.e.,* $\mu^*(E) = \mu_0(E)$ *for every* $E \in \mathcal{A}$.

(b) $\mathcal{A} \subseteq \Sigma$, *i.e., every set in* $\mathcal{A}$ *is* μ^**-measurable. Consequently,* $\mu(E) = \mu_0(E)$ *for every set* $E \in \mathcal{A}$.

(c) *If* ν *is any measure on* Σ *such that* $\nu|_{\mathcal{A}} = \mu_0$, *then* $\nu(E) \le \mu(E)$ *for all* $E \in \Sigma$, *with equality holding if* $\mu(E) < \infty$. *Furthermore, if* μ_0 *is* σ*-finite, then* $\nu = \mu$.

Proof. (a) Suppose that $E \in \mathcal{A}$. Then $\{E\}$ is a covering of E by a single set from $\mathcal{A}$, so $\mu^*(E) \le \mu_0(E)$. For the converse inequality, let $\{E_k\}$ be any countable collection of sets from $\mathcal{A}$ that covers E. Disjointize these sets by defining

$$F_1 \;=\; E \cap E_1 \qquad \text{and} \qquad F_n \;=\; E \cap \Bigl(E_n \setminus \bigcup_{k=1}^{n-1} E_k\Bigr), \quad n > 1.$$

Then $F_1, F_2, \dots \in \mathcal{A}$ and $\cup F_n = E \in \mathcal{A}$, so by the definition of premeasure and the fact that μ_0 is monotonic we have

$$\mu_0(E) \;=\; \sum_{n=1}^{\infty} \mu_0(F_n) \;\le\; \sum_{n=1}^{\infty} \mu_0(E_n).$$

This is true for every covering of E, so $\mu_0(E) \le \mu^*(E)$ and therefore μ^* agrees with μ_0 on $\mathcal{A}$.

(b) Suppose that $E \in \mathcal{A}$ and $A \subseteq X$. If we fix $\varepsilon > 0$, then there exists a countable covering $\{E_k\}$ of A by sets $E_k \in \mathcal{A}$ such that

$$\sum_{k=1}^{\infty} \mu_0(E_k) \;\le\; \mu^*(A) + \varepsilon.$$

Hence,

$$\begin{aligned}
\mu^*(A) \;&\le\; \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{C}}) && \text{(subadditivity)}\\
&\le\; \mu^*\Bigl(\Bigl(\bigcup_k E_k\Bigr) \cap E\Bigr) + \mu^*\Bigl(\Bigl(\bigcup_k E_k\Bigr) \cap E^{\mathrm{C}}\Bigr) && \text{(monotonicity)}\\
&=\; \mu^*\Bigl(\bigcup_k (E_k \cap E)\Bigr) + \mu^*\Bigl(\bigcup_k (E_k \cap E^{\mathrm{C}})\Bigr)\\
&\le\; \sum_{k=1}^{\infty} \mu^*(E_k \cap E) + \sum_{k=1}^{\infty} \mu^*(E_k \cap E^{\mathrm{C}}) && \text{(subadditivity)}\\
&=\; \sum_{k=1}^{\infty} \Bigl(\mu_0(E_k \cap E) + \mu_0(E_k \cap E^{\mathrm{C}})\Bigr) && \text{(part (a))}
\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \mu_0(E_k) \qquad \text{(finite additivity on } \mathcal{A})$$

$$\leq \mu^*(A) + \varepsilon.$$

Since this is true for every ε, we conclude that

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{C}}),$$

and therefore E is μ^*-measurable.

(c) Suppose that ν is any measure on Σ that extends μ_0, and fix $A \in \mathcal{A}$. If $\{E_k\}$ is a countable cover of A by sets $E_k \in \mathcal{A}$, then

$$\nu(A) \leq \sum_{k=1}^{\infty} \nu(E_k) = \sum_{k=1}^{\infty} \mu_0(E_k).$$

Since this is true for every covering, we conclude that $\nu(A) \leq \mu^*(A) = \mu(A)$.

Suppose in addition that $\mu(A) < \infty$. Then given $\varepsilon > 0$ we can find sets $E_k \in \mathcal{A}$ such that $\cup E_k \supseteq A$ and

$$\sum_k \mu_0(E_k) \leq \mu^*(A) + \varepsilon.$$

Set $E = \cup E_k$. Then

$$\begin{aligned} \mu(E) &\leq \sum_k \mu(E_k) && \text{(subadditivity)} \\ &= \sum_k \mu_0(E_k) && \text{(part (a))} \\ &\leq \mu^*(A) + \varepsilon \\ &= \mu(A) + \varepsilon && \text{(since } A \in \Sigma). \end{aligned}$$

Since all quantities are finite, we can rearrange and use additivity to conclude that that

$$\mu(E \backslash A) = \mu(E) - \mu(A) \leq \varepsilon.$$

Now, $\mathcal{A}$ is closed under finite unions, so $\cup_{k=1}^{N} E_k \in \mathcal{E}$ for every N. By continuity from below and the fact that μ and ν both extend μ_0, it follows that

$$\nu(E) = \lim_{N\to\infty} \nu\Big(\bigcup_{k=1}^{N} E_k\Big) = \lim_{N\to\infty} \mu_0\Big(\bigcup_{k=1}^{N} E_k\Big) = \lim_{N\to\infty} \mu\Big(\bigcup_{k=1}^{N} E_k\Big) = \mu(E).$$

Hence

$$\mu(A) \leq \mu(E) = \nu(E) = \nu(A) + \nu(E \backslash A) \leq \nu(A) + \mu(E \backslash A) \leq \nu(A) + \varepsilon.$$

Since ε is arbitrary, $\mu(A) = \nu(A)$.

Finally, suppose that μ_0 is σ-finite, i.e., we can write $X = \cup A_k$ where $\mu_0(A_k) < \infty$ for each k. By applying the disjointization trick, we can assume that the sets A_k are disjoint. Then since each A_k has finite measure, we have for any $E \in \Sigma$ that

$$\nu(E) \;=\; \sum_{k=1}^{\infty} \nu(E \cap A_k) \;=\; \sum_{k=1}^{\infty} \mu(E \cap A_k) \;=\; \mu(E). \qquad \diamondsuit$$

Additional Problems

2.24. Show that if μ^* is the outer measure induced from a premeasure μ_0, then every set $E \subseteq X$ that satisfies $\mu^*(E) = 0$ is μ^*-measurable. Conclude that $\mu = \mu|_\Sigma$ is a complete measure on (X, Σ).

2.25. Let $\mathcal{A}$ be an algebra on a set X. Let $\mathcal{A}_\sigma$ be the collection of countable unions of sets from $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets from $\mathcal{A}_\sigma$. Given a premeasure μ_0 on $\mathcal{A}$, prove the following statements.

(a) If $E \subseteq X$ and $\varepsilon > 0$, then there exists a set $A \in \mathcal{A}_\sigma$ such that $E \subseteq A$ and $\mu^*(A) \le \mu^*(E) + \varepsilon$, and there exists a set $H \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq H$ and $\mu^*(H) = \mu^*(E)$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists a set $H \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq H$ and $\mu^*(H \backslash E) = 0$.

(c) If μ_0 is σ-finite, then an arbitrary set $E \subseteq X$ is μ^*-measurable if and only if there exists a set $H \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq H$ and $\mu^*(H \backslash E) = 0$.

2.26. Let μ_0 be a premeasure on an algebra $\mathcal{A}$ of subsets of X. Suppose μ_0 is bounded ($\mu_0(X) < \infty$) and there is a set $A \subseteq X$ such that $\mu^*(A) = \mu_0(X)$. Show that $\mu^*(E) = \mu^*(E \cap A)$ for every μ^*-measurable set E.