
Measurable Functions

One of our main goals for the next few chapters is to develop a theory of integration with respect to an abstract measure. However, just as a measure need not be defined on every subset of X , we usually will not be able to integrate every possible function on X . We will need to restrict our attention to “measurable functions,” which we define and study in this chapter.

3.1 Measurable Functions

To motivate the definition of a measurable function, recall the definition of a continuous function. If f maps a metric space X (like \mathbb{R}^d , for example) into another metric space Y , then we can use the notion of distance to formulate continuity in terms of preservation of limits. However, there is a more fundamental definition of continuity that does depend on distance notions but rather on preservation of a topology. Given arbitrary topological spaces X and Y , a function $f: X \rightarrow Y$ is defined to be continuous if and only if *the inverse image of each open subset of Y is open in X* . That is, the following implication must hold:

$$U \text{ is an open subset of } Y \implies f^{-1}(U) \text{ is open subset of } X. \quad (3.1)$$

Just as the open sets are the “building blocks” of a topological space, the measurable sets are the building blocks of a measurable space. Substituting measurable sets for open sets in equation (3.1) leads to the following definition.

Definition 3.1 (Measurable Function). Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces. A function $f: X \rightarrow Y$ is (Σ_X, Σ_Y) -*measurable*, or simply *measurable* for short, if the following implication holds:

$$E \in \Sigma_Y \implies f^{-1}(E) \in \Sigma_X. \quad (3.2)$$

That is, the inverse image of every measurable subset of Y must be a measurable subset of X . \diamond

Notation 3.2. Sometimes we will write $f: (X, \Sigma_X) \rightarrow (Y, \Sigma_Y)$ to denote that f is a function from X to Y and Σ_X, Σ_Y are the σ -algebras on X and Y that we have in mind. This notation does not mean that f is measurable, but rather only means that Σ_X and Σ_Y are the σ -algebras that we should use in order to test whether f is a measurable function. \diamond

A measurable function *preserves structure* in the sense that the *inverse image* of a measurable set is measurable. However, measurability of a function does not tell us anything about *direct* images of sets. In general, a measurable function need not send a measurable set to a measurable set (see Lemma 1.63). This is analogous to the situation for continuity: A continuous function need not map an open set to an open set (consider the image of the interval $(0, 2\pi)$ under the function $f(x) = \sin x$). By imposing extra conditions on f we can sometimes make inferences about direct images of measurable sets. For example, Problem 4.1 gives some results in this direction.

Definition 3.1 can be difficult to implement directly. The next exercise gives a useful simplification of the test for measurability: Instead of showing that equation (3.2) holds for *every* set $E \in \Sigma_Y$, it suffices to only consider sets E that belong to a generating family \mathcal{E} for the σ -algebra Σ_Y .

Exercise 3.3. Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces, and suppose that Σ_Y is generated by a collection $\mathcal{E} \subseteq Y$, i.e., $\Sigma_Y = \Sigma(\mathcal{E})$. Show that the following two statements are equivalent.

- (a) $f: X \rightarrow Y$ is (Σ_X, Σ_Y) -measurable.
- (b) For each set $E \in \mathcal{E}$ we have $f^{-1}(E) \in \Sigma_X$. \diamond

As a corollary, we obtain an important result for measurability of functions that are defined on topological spaces.

Corollary 3.4. Let X, Y be topological spaces, and let $\mathcal{B}_X, \mathcal{B}_Y$ be the Borel σ -algebras on X and Y , respectively. Then

$$f: X \rightarrow Y \text{ is continuous} \implies f \text{ is } (\mathcal{B}_X, \mathcal{B}_Y)\text{-measurable.}$$

Proof. By definition, the open sets generate the Borel σ -algebra \mathcal{B}_Y . Also, since f is continuous we have by definition that $f^{-1}(U) \in \mathcal{B}_X$ for every open set $U \subseteq Y$. Exercise 3.3 therefore implies that f is $(\mathcal{B}_X, \mathcal{B}_Y)$ -measurable. \square

The converse of Corollary 3.4 will not usually be true, i.e., there will usually be measurable functions that are discontinuous. Also, it is important to note that Corollary 3.4 only applies when X and Y are topological spaces and we take the σ -algebras on X and Y to be the Borel σ -algebras.

Additional Problems

3.1. Let (X, Σ_X) be a measurable space, and let Y be any set. Given a function $f: X \rightarrow Y$, show that

$$\Sigma_Y = \{A \subseteq Y : f^{-1}(A) \in \Sigma_X\}$$

is a σ -algebra on Y .