3.2 Extended Real-Valued and Complex-Valued Functions

Many of the functions that we encounter in practice, and in particular most of the functions that we hope to integrate, are either real-valued, extended real-valued, or complex-valued. In other words, the space Y that we deal with most often is $\mathbb{R}$, $\overline{\mathbb{R}}$, or $\mathbb{C}$. We study measurability of these types of functions in this section.

Notation 3.5. In order to simplify the presentation of many statements, we will use the shorthand notations introduced in the opening section on General Notation. For example, if $f\colon X \to \overline{\mathbb{R}}$ is an extended real-valued function and $a \in \mathbb{R}$, then we use the shorthand $\{f > a\}$ to denote the set

$$\{f > a\} \;=\; \{x \in X : f(x) > a\} \;=\; f^{-1}(a, \infty].$$

Note that if f never takes the value ∞, then $\{f > a\} = f^{-1}(a, \infty)$. Other abbreviations are defined similarly. For example, if we also have a function $g\colon X \to \overline{\mathbb{R}}$, then we write

$$\{f \le g\} \;=\; \{x \in X : f(x) \le g(x)\}. \qquad \diamondsuit$$

Before getting into the details of measurability, we comment on the embeddings between the sets $\mathbb{R}$, $\overline{\mathbb{R}}$, and $\mathbb{C}$. Unfortunately, these three sets are not linearly ordered by inclusion. Although we have

$$\mathbb{R} \subseteq \overline{\mathbb{R}} \qquad \text{and} \qquad \mathbb{R} \subseteq \mathbb{C},$$

there is no inclusion between $\overline{\mathbb{R}}$ and $\mathbb{C}$. Every real-valued function is both an extended real-valued function and a complex-valued function, but an extended real-valued function *need not* be a complex-valued function. We do not allow a "complex infinity," so $\overline{\mathbb{R}}$ is not contained in $\mathbb{C}$. As a consequence, we have to state many definitions or theorems twice—once for extended real-valued functions and once for complex-valued functions. Each of these statements will include real-valued functions as a special case, although sometimes for clarity it is useful to explicitly state the real-valued setting as a third case. Usually, once we know the proper statement for one type of function it is clear what the statement should be for the others. Therefore we sometimes only deal with one case and leave the others as an exercise, or state hypotheses in a form such as "let f be a function on X (either extended real-valued or complex-valued)."

We introduce one more notational convention before proceeding, applicable to the case where f is an extended real-valued function on a measurable space (X, Σ) and we additionally have a measure μ that is defined on this space.

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Notation 3.6. Let (X, Σ, μ) be a measure space. We often encounter extended real-valued functions that are finite except on a set with μ-measure zero. If $f\colon X \to \overline{\mathbb{R}}$ is such that $\{f = \pm\infty\}$ is μ-measurable and

$$\mu\{f = \pm\infty\} \;=\; 0,$$

then we say that f *is finite* μ*-a.e.* ♢

As usual, we sometimes use the abbreviated phrase "f is finite a.e." if the measure μ is understood.

3.2.1 The Abstract Definition of Measurability

Let (X, Σ) be a measurable space, and let f be a function on X that takes values in $\mathbb{R}$, $\overline{\mathbb{R}}$, or $\mathbb{C}$. In order to define the meaning of measurability of f, we have to decide which σ-algebra that we will place on the codomains $\mathbb{R}$, $\overline{\mathbb{R}}$, or $\mathbb{C}$. In almost all circumstances, the appropriate algebra is the Borel σ-algebra, and therefore we take this as our default choice of σ-algebra on the codomains $\mathbb{R}$, $\overline{\mathbb{R}}$, or $\mathbb{C}$. Here is the restatement of Definition 3.1 for these particular codomains.

Definition 3.7. Let (X, Σ) be a measurable space.

(a) A function $f\colon X \to \mathbb{R}$ is *measurable* if it is $(\Sigma, \mathcal{B}_{\mathbb{R}})$-measurable.

(b) A function $f\colon X \to \overline{\mathbb{R}}$ is *measurable* if it is $(\Sigma, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

(b) A function $f\colon X \to \mathbb{C}$ is *measurable* if it is $(\Sigma, \mathcal{B}_{\mathbb{C}})$-measurable. ♢

That is, a function f is measurable if the inverse image of every Borel set in $\mathbb{R}$, $\overline{\mathbb{R}}$, or $\mathbb{C}$ (as appropriate) is a measurable subset of X. Of course, this definition is a bit premature since we have not yet defined the Borel σ-algebras on $\overline{\mathbb{R}}$ and $\mathbb{C}$. We will do this, and do it precisely, but for motivation we begin with the Borel algebra that we already know, which is the Borel σ-algebra on $\mathbb{R}$. We will give a much simpler equivalent formulation of measurability for real-valued functions, and once we see this it will be easy to move to the extended real-valued and complex-valued settings.

3.2.2 Real-Valued Functions

Let $f\colon X \to \mathbb{R}$ be a real-valued function on a set X. By Definition 3.7(a), measurability of f means $(\Sigma, \mathcal{B}_{\mathbb{R}})$-measurability. Stated explicitly, this means that $f\colon X \to \mathbb{R}$ is measurable if and only if

$$f^{-1}(E) \in \Sigma \text{ for every Borel set } E \subseteq \mathbb{R}. \tag{3.3}$$

Fortunately, Exercise 3.3 tells us that we do not need to test *every* Borel set in order to determine measurability—it is enough to just test sets contained in a generating family for the σ-algebra. Since the Borel σ-algebra on $\mathbb{R}$ is generated by the open sets (by definition), instead of taking every Borel set in equation (3.3) it is enough to just consider open sets E. Or we can use sets from any other generating family, such as the families given in Problem 2.5. In particular, each of the following is a generating family for $\mathcal{B}_{\mathbb{R}}$:

(a) $\mathcal{E}_1 = \{(a, \infty) : a \in \mathbb{R}\}$,
(b) $\mathcal{E}_2 = \{[a, \infty) : a \in \mathbb{R}\}$.
(c) $\mathcal{E}_3 = \{(-\infty, a) : a \in \mathbb{R}\}$.
(d) $\mathcal{E}_4 = \{(-\infty, a] : a \in \mathbb{R}\}$.

We can use sets from any of these families in equation (3.3) instead of using every Borel set. For example, choosing the family $\mathcal{E}_1$, we see that it is sufficient to just use sets of the form $E = (a, \infty)$ in equation (3.3). Since $f^{-1}(a, \infty) = \{f > a\}$, this tells us that

$$f \colon X \to \mathbb{R} \text{ is measurable} \quad \iff \quad \{f > a\} \in \Sigma \text{ for every } a \in \mathbb{R}.$$

By substituting other families that generate $\mathcal{B}_{\mathbb{R}}$, we obtain the following lemma.

Lemma 3.8. *Let (X, Σ) be a measurable space. Given $f \colon X \to \mathbb{R}$, the following statements are equivalent.*

(a) *f is measurable.*
(b) *$\{f > a\} \in \Sigma$ for every $a \in \mathbb{R}$.*
(c) *$\{f \geq a\} \in \Sigma$ for every $a \in \mathbb{R}$.*
(d) *$\{f < a\} \in \Sigma$ for every $a \in \mathbb{R}$.*
(e) *$\{f \leq a\} \in \Sigma$ for every $a \in \mathbb{R}$.* $\diamondsuit$

3.2.3 Extended Real-Valued Functions

In order to characterize measurability of extended real-valued functions, we need to understand the Borel σ-algebra on $\overline{\mathbb{R}}$, or we at least need to know a generating set for $\mathcal{B}_{\overline{\mathbb{R}}}$. We will give a detailed discussion of $\mathcal{B}_{\overline{\mathbb{R}}}$ in a moment, but for most purposes it is sufficient just to know that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the family of intervals of the form $(a, \infty]$. This seems quite reasonable in light of the fact that the family of intervals (a, ∞) generates the Borel σ-algebra on $\mathbb{R}$. So for now let us simply take this fact as an axiom, and see what it implies about measurability of extended real-valued functions.

Axiom 3.9. Each of the following families of subsets of $\overline{\mathbb{R}}$ generates the Borel σ-algebra on $\overline{\mathbb{R}}$:

(a) $\mathcal{E}_1 = \{(a, \infty] : a \in \mathbb{R}\}$,
(b) $\mathcal{E}_2 = \{[a, \infty] : a \in \mathbb{R}\}$.
(c) $\mathcal{E}_3 = \{[-\infty, a) : a \in \mathbb{R}\}$.
(d) $\mathcal{E}_4 = \{[-\infty, a] : a \in \mathbb{R}\}$. $\diamondsuit$

For an extended real-valued function,

$$\{f > a\} \;=\; \{x \in X : f(x) > a\} \;=\; f^{-1}(a, \infty],$$
$$\{f \ge a\} \;=\; \{x \in X : f(x) \ge a\} \;=\; f^{-1}[a, \infty],$$

and so forth. Therefore, Axiom 3.9 immediately implies the following characterization of measurability of extended real-valued functions.

Lemma 3.10. *Let (X, Σ) be a measurable space. Given $f \colon X \to \overline{\mathbb{R}}$, the following statements are equivalent.*

(a) f *is measurable.*

(b) $\{f > a\} \in \Sigma$ *for every* $a \in \mathbb{R}$.

(c) $\{f \ge a\} \in \Sigma$ *for every* $a \in \mathbb{R}$.

(d) $\{f < a\} \in \Sigma$ *for every* $a \in \mathbb{R}$.

(e) $\{f \le a\} \in \Sigma$ *for every* $a \in \mathbb{R}$. $\diamondsuit$

Note the similarity between Lemma 3.8 and Lemma 3.10! The only difference is that the functions considered in Lemma 3.10 can take the values $\pm\infty$ in addition to real values.

In practice, Lemma 3.10 is what we usually need to know about measurability of extended real-valued functions. However, for the interested reader we give a more detailed discussion of the Borel σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$, and explain why Axiom 3.9 holds. This discussion can be skipped without much loss, but the exercises in the discussion do give a good insight into the similarities and differences between topologies and σ-algebras. A brief review of topologies, generating families, and bases for a topology can be found in Appendix A.

By definition, the Borel σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ is the smallest σ-algebra on $\overline{\mathbb{R}}$ that contains all of the open subsets of $\overline{\mathbb{R}}$. So the real issue in defining $\mathcal{B}_{\overline{\mathbb{R}}}$ is this question: What are the open subsets of $\overline{\mathbb{R}}$? Or, in more technical jargon: What is the topology of $\overline{\mathbb{R}}$? Of course, there are many possible topologies on $\overline{\mathbb{R}}$, but since we are thinking of $\overline{\mathbb{R}}$ as being an extension of $\mathbb{R}$, we seek a topology on $\overline{\mathbb{R}}$ that extends the topology of $\mathbb{R}$ in a natural way. Every subset of $\mathbb{R}$ that is open with respect to the usual topology on $\mathbb{R}$ should still be an open subset of $\overline{\mathbb{R}}$. Also, sets like $(a, \infty]$ should be open, but we do not want a singleton like $\{\infty\}$ to be open.

Now, the topology $\mathcal{T}_{\mathbb{R}}$ of $\mathbb{R}$ is generated by the open intervals (a, b). In fact, every open subset of $\mathbb{R}$ can be written as a union of open intervals (a, b), so the collection $\beta_{\mathbb{R}}$ of all open intervals (a, b) is called a *base* for the topology on $\mathbb{R}$. The next definition extends this idea to $\overline{\mathbb{R}}$ by combining the infinite intervals $(a, \infty]$ and $[-\infty, b)$ with the finite open intervals (a, b).

Definition 3.11 (Topology of $\overline{\mathbb{R}}$). A subset of $\overline{\mathbb{R}}$ is open if it can be written as a union of intervals or the form (a, b), $(a, \infty]$, or $[-\infty, b)$, where a, b belong to $\mathbb{R}$. We set

$$\beta_{\overline{\mathbb{R}}} = \{(a,b) : a < b\} \cup \{(a,\infty] : a \in \mathbb{R}\} \cup \{[-\infty,b) : b \in \mathbb{R}\}.$$

The topology of $\overline{\mathbb{R}}$ is the collection $\mathcal{T}_{\overline{\mathbb{R}}}$ containing all of open subsets of $\overline{\mathbb{R}}$, i.e., $\mathcal{T}_{\overline{\mathbb{R}}}$ is the set of all possible unions of elements of $\beta_{\overline{\mathbb{R}}}$. ♢

In the language of topology, because every open subset of $\overline{\mathbb{R}}$ is a union of elements of $\beta_{\overline{\mathbb{R}}}$, the collection $\beta_{\overline{\mathbb{R}}}$ called a *base* for the topology $\mathcal{T}_{\overline{\mathbb{R}}}$. Of course, we jumping a little ahead of ourselves here, because we have not yet shown that $\mathcal{T}_{\overline{\mathbb{R}}}$ satisfies the requirements of a topology. This is done in the next exercise, which also shows that the open subsets of $\overline{\mathbb{R}}$ can be written as countable unions of elements of $\beta_{\overline{\mathbb{R}}}$.

Exercise 3.12. (a) Show that the family $\beta_{\overline{\mathbb{R}}}$ is closed under finite intersections.

(b) Show that $\mathcal{T}_{\overline{\mathbb{R}}}$ is a topology on $\overline{\mathbb{R}}$, i.e., it is nonempty, closed under finite intersections, and closed under arbitrary unions.

(c) By definition, a set $U \subseteq \overline{\mathbb{R}}$ is open if and only if U can be written as a union of intervals of the form (a,b), $(a,\infty]$, or $[-\infty,b)$. Show that U is open if and only if it can be written as a *countable* union of such intervals. ♢

Remark 3.13. (a) By construction, every open subset of $\mathbb{R}$ is an open subset of $\overline{\mathbb{R}}$. In particular, the real line $\mathbb{R}$ is an open subset of $\overline{\mathbb{R}}$.

(b) If U is an open subset of $\overline{\mathbb{R}}$ then $U \cap \mathbb{R}$ is a union of intervals of the form (a,b), (a,∞), or $(-\infty,b)$, and therefore is an open subset of $\mathbb{R}$. However, the converse need not hold, i.e., $U \cap \mathbb{R}$ may be an open subset of $\mathbb{R}$ even though U is not an open subset of $\overline{\mathbb{R}}$. For example, $U = (0,1) \cup \{\infty\}$ is not an open subset of $\overline{\mathbb{R}}$, but its intersection with $\mathbb{R}$ is the open interval (a,b). ♢

So, we have defined the topology of $\overline{\mathbb{R}}$, and we have determined a generating family (in fact, a base) for this topology. By definition, the Borel σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ is the smallest σ-algebra on $\overline{\mathbb{R}}$ that contains all of the open subsets of $\overline{\mathbb{R}}$. The next exercise gives some generating families for $\mathcal{B}_{\overline{\mathbb{R}}}$, and thereby justifies Axiom 3.9.

Exercise 3.14. Show that $\mathcal{B}_{\overline{\mathbb{R}}}$ is the smallest σ-algebra on $\overline{\mathbb{R}}$ that contains all intervals of the form $(a,\infty]$. That is, show that if we set

$$\mathcal{E} = \{(a,\infty] : a \in \mathbb{R}\},$$

then $\mathcal{B}_{\overline{\mathbb{R}}} = \Sigma(\mathcal{E})$. Also show that each of the other families given in Axiom 3.9 generate $\mathcal{B}_{\overline{\mathbb{R}}}$. ♢

Finally, the next exercise shows that the Borel subsets of $\overline{\mathbb{R}}$ are simply the Borel subsets of $\mathbb{R}$ with $\pm\infty$ possibly adjoined. In particular, every set that is a Borel set in $\mathbb{R}$ is a Borel set in $\overline{\mathbb{R}}$.

Exercise 3.15. Given $E \subseteq \overline{\mathbb{R}}$, show that the following statements are equivalent.

(a) E is a Borel set in $\overline{\mathbb{R}}$.

(b) $E \cap \mathbb{R}$ is a Borel set in $\mathbb{R}$.

(c) E has one of the following forms:

$$A, \quad A \cup \{\infty\}, \quad A \cup \{-\infty\}, \quad A \cup \{-\infty, \infty\},$$

where A is a Borel set in $\mathbb{R}$. $\diamondsuit$

Another approach to constructing the Borel σ-algebra on $\overline{\mathbb{R}}$ is to make $\overline{\mathbb{R}}$ into a metric space by defining the distance between points $x, y \in \overline{\mathbb{R}}$ to be

$$\mathrm{d}(x,y) \;=\; |\arctan(x) - \arctan(y)|,$$

where we take $\arctan(\infty) = \pi/2$ and $\arctan(-\infty) = -\pi/2$. This produces exactly the same σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$.

3.2.4 Complex-Valued Functions

Now we consider complex-valued functions on X (which includes real-valued functions as a special case). By definition, the Borel σ-algebra on the complex plane is the σ-algebra that is generated by the open subsets of $\mathbb{C}$. The topology on $\mathbb{C}$ is simply the usual Euclidean topology of a plane, i.e., we identify $\mathbb{C}$ with $\mathbb{R}^2$ and identify the open subsets of $\mathbb{C}$ with the open subsets of $\mathbb{R}^2$. For example, the "open rectangle" $\{x+iy : a < x < b, c < y < d\}$ is an open subset of $\mathbb{C}$. Since the open sets generate the Borel σ-algebra $\mathcal{B}_{\mathbb{C}}$, it follows that a function $f\colon X \to \mathbb{C}$ is measurable if and only if

$$f^{-1}(U) \in \Sigma \text{ for every open set } U \subseteq \mathbb{C}.$$

However, if we like we can use other generating families for $\mathcal{B}_{\mathbb{C}}$ to test for measurability. For example, Problem 2.6 gives several families that generate the Borel σ-algebra on $\mathbb{R}^2$, and by making the appropriate identifications this also gives us generating families for $\mathcal{B}_{\mathbb{C}}$. In particular, it is often convenient to work with the generating family $\mathcal{E}$ that consists of all of the horizontal or vertical open strips

$$\bigl\{x+iy : a < x < b,\, y \in \mathbb{R}\bigr\} \qquad \text{and} \qquad \bigl\{x+iy : x \in \mathbb{R},\, c < y < d\bigr\}.$$

A function $f\colon X \to \mathbb{C}$ is measurable if and only if the inverse image of any one of these strips is a measurable subset of X. This is precisely the family to use to solve the next exercise.

Exercise 3.16. Let (X, Σ) be a measurable space. Given $f\colon X \to \mathbb{C}$, let f_r and f_i be the real and imaginary parts of f, i.e., $f = f_r + i f_i$ where f_r, f_i are real-valued. Show that

$$f \text{ is measurable} \quad \Longleftrightarrow \quad f_r \text{ and } f_i \text{ are both measurable.} \qquad \diamondsuit$$

Additional Problems

3.2. Let (X, Σ) be a measure space and fix $f\colon X \to \overline{\mathbb{R}}$. Show that if A is a dense subset of $\mathbb{R}$, then f is measurable if and only if $\{f > a\}$ is measurable for each $a \in A$.

3.3. Let (X, Σ) be a measure space. Show that if $f\colon X \to \overline{\mathbb{R}}$ is measurable then $\{f = a\}$ is measurable in X for every $a \in \overline{\mathbb{R}}$, but the converse statement can fail.

3.4. Let X be a set. Characterize the functions $f\colon X \to \overline{\mathbb{R}}$ that are measurable with respect to the following σ-algebras on X.

(a) $\Sigma = \mathcal{P}(X)$.

(b) $\Sigma = \{\emptyset, X\}$.

(c) Σ is the σ-algebra constructed in Exercise 2.5.

3.5. Let μ be a bounded measure on a measure space (X, Σ), i.e., $\mu(X) < \infty$. Assume that $f\colon X \to [0, \infty]$ is measurable. Show that f is finite μ-a.e. if and only if for every $\varepsilon > 0$ there exists an $M > 0$ such that $\mu\{f > M\} < \varepsilon$.