3.3 Measurable Functions on the Domain $\mathbb{R}^d$

In the preceding section we considered extended real-valued and complex-valued functions whose domain was a generic measurable space (X, Σ). Now we consider the special case of functions defined on the domain $\mathbb{R}^d$, or on subsets of $\mathbb{R}^d$.

3.3.1 Extended Real-Valued Functions on $\mathbb{R}^d$

Let $f \colon \mathbb{R}^d \to \overline{\mathbb{R}}$ be an extended real-valued function whose domain is all of $\mathbb{R}^d$. We have declared that measurability of f means $(\Sigma, \mathcal{B}_{\overline{\mathbb{R}}})$-measurability, i.e., we place the Borel σ-algebra on the codomain $\overline{\mathbb{R}}$. However, this still leaves us with the question of which σ-algebra we should place on the domain $\mathbb{R}^d$. Since we placed the Borel σ-algebra on the codomain, we might expect that we will also place the Borel σ-algebra on the domain $\mathbb{R}^d$, but there is no reason why we are required to do so. Indeed, in many circumstances it is more natural to consider the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ on $\mathbb{R}^d$, and the next definition states that this is our default choice.

Definition 3.17. Let $f \colon \mathbb{R}^d \to \overline{\mathbb{R}}$ be given.

(a) We say that f is *Borel measurable* if f is $(\mathcal{B}_{\mathbb{R}^d}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. By Lemma 3.10, f is Borel measurable if and only if

$$\forall\, a \in \mathbb{R}, \quad \{f > a\} \text{ is a Borel set in } \mathbb{R}^d.$$

(b) We say that f is *Lebesgue measurable* if f is $(\mathcal{L}_{\mathbb{R}^d}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. By Lemma 3.10, f is Lebesgue measurable if and only if

$$\forall\, a \in \mathbb{R}, \quad \{f > a\} \text{ is a Lebesgue measurable set in } \mathbb{R}^d.$$

If we write "$f \colon \mathbb{R}^d \to \overline{\mathbb{R}}$ is measurable" without qualification, then we mean that f is Lebesgue measurable. ♢

Since every Borel set is Lebesgue measurable, we have the following implication:

$$f \text{ is Borel measurable} \quad \Longrightarrow \quad f \text{ is Lebesgue measurable.}$$

However, the next example shows that the converse implication does not hold in general.

Example 3.18. Let E be a subset of $\mathbb{R}^d$, and consider the characteristic function χ_E of E. Given $a \in \mathbb{R}$, we have

$$\{\chi_E > a\} \;=\; \begin{cases} \emptyset, & a \ge 1, \\ E, & 0 \le a < 1, \\ \mathbb{R}^d, & a < 0. \end{cases}$$

Hence χ_E is a Lebesgue measurable function on $\mathbb{R}^d$ if and only if E is a Lebesgue measurable subset of $\mathbb{R}^d$, and χ_E is Borel measurable if and only if E is a Borel set in $\mathbb{R}^d$. ◇

Here are some additional examples of measurable functions.

Lemma 3.19. *If $f\colon \mathbb{R} \to \mathbb{R}$ is monotone increasing, then f is Borel measurable (and hence is also Lebesgue measurable).*

Proof. Fix $a \in \mathbb{R}$. Since f is monotone increasing, $\{f > a\}$ is either an interval of the form (x, ∞), or it is an interval of the form $[x, \infty)$. Each of these are Borel sets, so f is Borel measurable. □

Lemma 3.20. *If $f\colon \mathbb{R}^d \to \mathbb{R}$ is continuous, then f is Borel measurable (and hence is also Lebesgue measurable).*

Proof. Fix $a \in \mathbb{R}$. Since the interval (a, ∞) is an open subset of $\mathbb{R}$ and f is continuous, the inverse image $f^{-1}(a, \infty)$ is open in $\mathbb{R}^d$ and hence is a Borel set. Since $\{f > a\} = f^{-1}(a, \infty)$, this shows that f is Borel measurable. □

3.3.2 Complex-Valued Functions on $\mathbb{R}^d$

Now we turn to the complex-valued case. It may seem odd at first to consider *complex-valued* functions on the *real Euclidean space* $\mathbb{R}^d$, but such functions arise quite often in practice. For example, given a fixed *frequency* $\xi \in \mathbb{R}$, the *complex exponential function* $e_\xi\colon \mathbb{R} \to \mathbb{C}$ is defined by

$$e_\xi(x) \;=\; e^{2\pi i \xi x}, \quad x \in \mathbb{R}.$$

The complex exponentials play an important role in many areas of mathematics and engineering, including harmonic analysis and signal processing. In higher dimensions, we fix $\xi \in \mathbb{R}^d$ and define $e_\xi\colon \mathbb{R}^d \to \mathbb{C}$ by

$$e_\xi(x) \;=\; e^{2\pi i \xi \cdot x}, \quad x \in \mathbb{R}^d,$$

where $\xi \cdot x$ is the usual dot product of vectors in $\mathbb{R}^d$.

We make an entirely analogous version of Definition 3.17 for complex-valued functions.

Definition 3.21. Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be given.

(a) We say that f is *Borel measurable* if f is $(\mathcal{B}_{\mathbb{R}^d}, \mathcal{B}_{\mathbb{C}})$-measurable. By Exercise 3.16, f is Borel measurable if and only if its real and imaginary parts are real-valued Borel measurable functions.

(b) We say that f is *Lebesgue measurable* if f is $(\mathcal{L}_{\mathbb{R}^d}, \mathcal{B}_{\mathbb{C}})$-measurable. By Exercise 3.16, f is Lebesgue measurable if and only if its real and imaginary parts are real-valued Lebesgue measurable functions.

If we write "$f\colon \mathbb{R}^d \to \mathbb{C}$ is measurable" without qualification, then we mean that f is Lebesgue measurable. ♢

Example 3.22. Suppose that $f\colon \mathbb{R}^d \to \mathbb{C}$ is continuous. Then the inverse image of any open subset of $\mathbb{C}$ is open in $\mathbb{R}^d$ and hence is a Borel set. Since the open sets generate the Borel σ-algebra, this tells us that f is Borel measurable, and hence is also Lebesgue measurable. Alternatively, we can simply observe that the real and imaginary parts of a continuous complex-valued function are continuous, and every continuous real-valued function is measurable by Lemma 3.20. ♢

By identifying $\mathbb{R}^2$ with $\mathbb{C}$, Definition 3.21 also covers the case of functions that map the complex plane into itself. In particular, Example 3.22 tells us that every continuous function $f\colon \mathbb{C} \to \mathbb{C}$ is Borel measurable. The next exercise constructs a square-root function on $\mathbb{C}$ that is not continuous, but is still Borel measurable. One way to solve this exercise is to show directly that the inverse image of every open subset of $\mathbb{C}$ is a Borel set in $\mathbb{C}$ (or, it suffices to just consider sets in a generating family instead of every open set).

Exercise 3.23. Define a square root function $Sz = z^{1/2}$ on $\mathbb{C}$ by

$$(re^{i\theta})^{1/2} \;=\; r^{1/2}e^{i\theta/2}, \qquad r > 0,\ 0 \le \theta < 2\pi.$$

Show that $S\colon \mathbb{C} \to \mathbb{C}$ is Borel measurable (though not continuous). ♢

3.3.3 Functions on Subsets of $\mathbb{R}^d$

Suppose we are given an extended real-valued function f whose domain is only a subset D of $\mathbb{R}^d$. To define measurability of this function f we simply need to modify Definition 3.17 appropriately. Focusing on Lebesgue measurability, it seems reasonable to say that f is Lebesgue measurable if and only if $\{f > a\}$ is a Lebesgue measurable subset of D for every $a \in \mathbb{R}$. This is exactly what we do, but the technical issue is that we have not yet defined exactly what "Lebesgue measurable subset of D" means. We address this issue now.

By Problem 2.2, we can define a σ-algebra on D by intersecting the Lebesgue measurable subsets of $\mathbb{R}^d$ with D. Precisely,

$$\mathcal{L}_D \;=\; \bigl\{E \cap D : E \in \mathcal{L}_{\mathbb{R}^d}\bigr\} \tag{3.4}$$

defines a σ-algebra of subsets of D. Unfortunately, if D is not itself Lebesgue measurable, then the sets in $\mathcal{L}_D$ need not be Lebesgue measurable subsets of $\mathbb{R}^d$. On the other hand, if D is a Lebesgue measurable subset of $\mathbb{R}^d$ then so is every set in $\mathcal{L}_D$, and we have the following characterization of $\mathcal{L}_D$.

Exercise 3.24. Show that if D is a Lebesgue measurable subset of $\mathbb{R}^d$, then

$$\mathcal{L}_D = \{E \in \mathcal{L}_{\mathbb{R}^d} : E \subseteq D\}. \qquad \diamondsuit \tag{3.5}$$

Thus, as long as D is Lebesgue measurable, the Lebesgue measurable subsets of D are exactly what they should be—they are the Lebesgue measurable subsets of $\mathbb{R}^d$ that are contained in D.

Definition 3.25. If D is a Lebesgue measurable set in $\mathbb{R}^d$, then the family $\mathcal{L}_D$ is called the *Lebesgue σ-algebra on D*, and the elements of $\mathcal{L}_D$ are the *Lebesgue measurable subsets of D*. $\diamondsuit$

If D is not Lebesgue measurable then we can still define $\mathcal{L}_D$ by equation (3.4), but in this case the equality given in equation (3.5) will not hold. In most circumstances it is not very useful to work with functions defined on a nonmeasurable domain.

Consequently, if D is Lebesgue measurable then we can characterize the Lebesgue measurable functions on D as follows.

Lemma 3.26. *Let D be a Lebesgue measurable subset of $\mathbb{R}^d$.*

(a) *An extended real-valued function $f\colon D \to \overline{\mathbb{R}}$ is Lebesgue measurable if and only if $\{f > a\} = \{x \in D : f(x) > a\}$ is Lebesgue measurable for each $a \in \mathbb{R}$.*

(b) *A complex-valued function $f\colon D \to \mathbb{C}$ is Lebesgue measurable if and only if its real and imaginary parts are Lebesgue measurable.* $\diamondsuit$

Similar remarks and definitions apply if we are interested in Borel measurability instead of Lebesgue measurability, and we leave it to the reader to formulate analogues of the definitions above for Borel measurability.

Additional Problems

3.6. Let D be a measurable subset of $\mathbb{R}^d$, and assume that $f\colon D \to \overline{\mathbb{R}}$ is measurable. Extend f to $\mathbb{R}^d$ by defining $f(x) = 0$ for $x \notin D$, and show that this extended function f is measurable.

3.7. Let $D \subseteq \mathbb{R}^d$ be measurable with $|D| < \infty$. Let $f_k\colon D \to \overline{\mathbb{R}}$ be measurable functions, and suppose that for each $x \in D$ we have

$$M_x = \sup_{k \in \mathbb{N}} |f_k(x)| < \infty.$$

Show that for each $\varepsilon > 0$, there exists a closed set $F \subseteq E$ and a finite constant M such that $|E \backslash F| < \varepsilon$ and $|f_k(x)| \le M$ for all k and all $x \in F$.