3.4 Functions Equal Almost Everywhere

If (X, Σ, μ) is a complete measure space, then every subset of a null set is measurable. This gives us the following important lemma, which states that if we change the values of a measurable function f on a set with μ-measure zero, then the new function is still measurable.

Lemma 3.27. *Let (X, Σ, μ) be a complete measure space, and let f be a measurable function on X (either extended real-valued or complex-valued). If g is a function on X such that $f = g$ μ-a.e., then g is measurable.*

Proof. Assume f and g are extended real-valued functions, and $f = g$ μ-a.e. This means that the set $Z = \{f \neq g\}$ is measurable and $\mu(Z) = 0$.

Fix $a \in \mathbb{R}$. By hypothesis, the set $A = \{f < a\}$ is measurable, and our goal is to show that $B = \{g > a\}$ is also measurable. However, $A \backslash B \subseteq Z$ and μ is complete, so we know that $A \backslash B$ is measurable. Similarly, $B \backslash A$ is measurable, so

$$B \;=\; A \cup (B \backslash A) \setminus (A \backslash B)$$

is measurable, and therefore g is a measurable function.

The complex case follows by splitting into real and imaginary parts. □

Here is a useful corollary for functions on $\mathbb{R}^d$. Although we do not explicitly specify a measure on $\mathbb{R}^d$ in this corollary, following Notation 2.16 we implicitly assume that the measure is our default choice of Lebesgue measure.

Corollary 3.28. *Let $f \colon \mathbb{R}^d \to \mathbb{C}$ be given. If there is a continuous function $g \colon \mathbb{R}^d \to \mathbb{C}$ such that $f = g$ almost everywhere, then f is measurable.* ♢

For example, the characteristic function χ_C of the Cantor set equals the zero function almost everywhere, so χ_C is Lebesgue measurable.

In summary, as far as measurability goes (and in many other contexts as well), when μ is complete we can usually ignore sets of measure zero. Here are two situations where this can be important.

First, suppose that (X, Σ, μ) is a measure space and $f \colon X \to \overline{\mathbb{R}}$ is an extended real-valued function on X. If f is measurable and finite μ-a.e., then the set on which f takes the values $\pm\infty$ has measure zero. By changing the values of f on a null set, we can therefore create a new function g that is measurable, equals f almost everywhere, and is finite at every point of X.

Second, if we are given a function that is defined on all of X except for a subset Z that has measure zero then we can assign any values we like to $f(x)$ for $x \in Z$ without affecting the measurability of f. In this way, functions defined on $X \backslash Z$ can usually be implicitly assumed to be defined on all of X.

Notation 3.29. Let (X, Σ, μ) be a measure space. If Z is a μ-null set, then we often say that a function f whose domain is $X \backslash Z$ is *defined almost everywhere on X*. ♢

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For example, the function $f(x) = 1/x$ is defined almost everywhere on $\mathbb{R}$ with respect to Lebesgue measure, because Lebesgue measure is complete and the singleton $\{0\}$ has Lebesgue measure zero. We can regard f as a function on $\mathbb{R}$ by assigning any value we like to $f(0)$, and this will not have any impact on whether f is measurable or not. However, we do not yet know whether f is measurable. For example, if we knew that there was a continuous function that equaled f almost everywhere, then we could apply Corollary 3.28 and conclude that f is measurable. However, although f is continuous at all but one point (hence is *continuous almost everywhere*), there is no continuous function that equals f almost everywhere. Still, it is not too hard to check that this f is indeed measurable.

Exercise 3.30. Show directly that $f(x) = 1/x$ is measurable. ♢

More generally, Problem 3.15 asks for a proof that every function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous almost everywhere on $\mathbb{R}$ is measurable.