3.5 Compositions of Measurable Functions

In the next few sections we will examine how measurability behaves with respect to the usual operations on functions, such as composition, addition, multiplication, limits, and so forth. We will see that measurability is preserved in most situations, although there are certain circumstances where we need to exhibit a little care.

We will treat compositions in this section. The next lemma shows that measurability is preserved under compositions if we have the appropriate match in the σ-algebras placed on our spaces.

Lemma 3.31. *Let (X, Σ_X), (Y, Σ_Y), and (Z, Σ_Z) be measurable spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are measurable with respect to the given σ-algebras, then $g \circ f\colon X \to Z$ is measurable.*

Proof. We simply have to verify that if E is a measurable subset of Z then its inverse image under $g \circ f$ is a measurable subset of X. This follows from the individual measurability of f and g, for if $E \in \Sigma_Z$ then $g^{-1}(E) \in \Sigma_Y$ since g is measurable, and hence

$$(g \circ f)^{-1}(E) \;=\; f^{-1}\big(g^{-1}(E)\big) \;\in\; \Sigma_X$$

since f is measurable. □

To emphasize, the hypotheses of Lemma 3.31 are that

$$f \text{ is } (\Sigma_X, \Sigma_Y)\text{-measurable} \qquad \text{and} \qquad g \text{ is } (\Sigma_Y, \Sigma_Z)\text{-measurable.}$$

Despite the naturalness of these hypotheses, we must be careful when attempting to apply Lemma 3.31 to extended real-valued or complex-valued functions. For example, suppose that we have a measurable real-valued function $f\colon X \to \mathbb{R}$, and we know that $g\colon \mathbb{R} \to \mathbb{R}$ is also measurable. Following the conventions of Definition 3.17, if we write "g is measurable" without qualification, it means that g is *Lebesgue measurable*. Thus, we are assuming that

$$f \text{ is } (\Sigma, \mathcal{B}_{\mathbb{R}})\text{-measurable} \qquad \text{and} \qquad g \text{ is } (\mathcal{L}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})\text{-measurable.}$$

We do not have the required matchup in σ-algebras that is required in order to apply Lemma 3.31. In general, the composition of a measurable function $f\colon X \to \mathbb{R}$ with a measurable function $g\colon \mathbb{R} \to \mathbb{R}$ *need not be measurable*, the basic problem being that if $E \in \mathcal{B}_{\mathbb{R}}$ then we only know that $g^{-1}(E)$ is Lebesgue measurable, whereas we need to know that $g^{-1}(E)$ is Borel measurable in order to conclude that $f^{-1}(g^{-1}(E))$ is measurable in X (see Problem 3.10). Of course, in order to fix this problem we just need to impose a slightly stronger hypothesis on g.

Lemma 3.32. *Let (X,Σ) be a measurable space. If $f\colon X\to\mathbb{R}$ is measurable and $g\colon\mathbb{R}\to\mathbb{R}$ is Borel measurable, then $g\circ f\colon X\to\mathbb{R}$ is measurable.* $\diamondsuit$

For example, Lemma 3.20 tells us that every continuous function $g\colon\mathbb{R}\to\mathbb{R}$ is Borel measurable, so we have many ways to obtain new measurable functions from a given measurable function. We state this explicitly as follows.

Corollary 3.33. *Let (X,Σ) be a measurable space. If $f\colon X\to\mathbb{R}$ is measurable and $g\colon\mathbb{R}\to\mathbb{R}$ is continuous, then $g\circ f\colon X\to\mathbb{R}$ is measurable.* $\diamondsuit$

The next corollary gives some of the most common applications of this idea (the notations f^+, f^- for the positive and negative parts of f were introduced in the opening section on General Notation).

Corollary 3.34. *Assume (X,Σ) is a measurable space and $f\colon X\to\mathbb{R}$ is a measurable function. Then each of the following are also real-valued measurable functions on X:*

(a) $|f|$,

(b) $|f|^p$ *for each real number* $p>0$,

(c) $f^+(x)=\max\{f(x),0\}$,

(d) $f^-(x)=\max\{-f(x),0\}$,

(e) $e^{f(x)}$, $\sin f(x)$, $\cos f(x)$.

Proof. To prove statement (b), fix any $p>0$. The function $g(x)=|x|^p$ is a continuous mapping of $\mathbb{R}$ into itself, so if f is measurable then so is $|f|=g\circ f$. The other statements follow similarly. $\square$

Analogues of these results can be easily formulated for complex-valued functions, e.g., see Problem 3.8. However, there is a technicality that can be problematic when dealing with extended real-valued functions. Often, we have a measurable function $f\colon X\to\overline{\mathbb{R}}$ but we wish to compose it with a continuous or Borel measurable function g that is defined on $\mathbb{R}$ rather than $\overline{\mathbb{R}}$. The next exercise shows that as long as f does not take the values $\pm\infty$ on a set of positive measure, and as long as our measure is complete, this does not pose a problem.

Exercise 3.35. Let $f\colon X\to\overline{\mathbb{R}}$ be a measurable function on a complete measure space (X,Σ,μ), and assume f is finite μ-a.e.

(a) Show that if $g\colon\mathbb{R}\to\mathbb{R}$ is Borel measurable, then $g\circ f$ is defined μ-a.e. on X and is measurable.

(b) Show that $|f|^p$ is measurable for each real number $p>0$, and both f^+ and f^- are measurable. $\diamondsuit$

The "finite almost everywhere" assumption of Exercise 3.35 is a commonly encountered hypothesis. Precisely (see Notation 3.6), it means that $\{f = \pm\infty\}$ is a μ-measurable set that has μ-measure zero. Often we deal with a measure μ that is induced from an outer measure μ^* (e.g., Lebesgue measure). In this case all sets Z that satisfy $\mu^*(Z) = 0$ are measurable, so f is finite μ-a.e. if and only if $\mu^*\{f = \pm\infty\} = 0$.

Sometimes we can remove the finite almost everywhere requirement. The next exercise can be done directly, or by thinking about what it means for a function $g\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ to be continuous.

Exercise 3.36. Given a measurable function $f\colon X \to \overline{\mathbb{R}}$ on a measurable space (X, Σ), show that $|f|^p$ is measurable for each real number $p > 0$, and both f^+ and f^- are measurable. $\diamondsuit$

Additional Problems

3.8. Let (X, Σ) be a measure space. Show that if $f\colon X \to \mathbb{C}$ is measurable, then so is $|f|^p$ for each real number $p > 0$.

3.9. Give an example of a nonmeasurable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $|f|$ is Lebesgue measurable.

3.10. Give an example of a Lebesgue measurable function $f\colon \mathbb{R} \to \mathbb{R}$ and a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is not Lebesgue measurable.

3.11. (a) Given $z \in \mathbb{C}$, define

$$\operatorname{sgn} z = \begin{cases} \dfrac{z}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Show that $\operatorname{sgn}\colon \mathbb{C} \to \mathbb{C}$ is Borel measurable.

(b) Show that if $f\colon (X, \Sigma) \to \mathbb{C}$ is a measurable function, then we can write f in *polar form* as $f = (\operatorname{sgn} f)\,|f|$, and both $\operatorname{sgn} f$ and $|f|$ are measurable.