3.6 Arithmetic with Functions

Now we will see how measurability of functions behaves with respect to the usual operations of arithmetic. We will focus on extended real-valued functions. Most of the results then carry over immediately to complex-valued functions by breaking into real and imaginary parts, and at the end of this section we state an exercise that summarizes the extension to the complex case.

3.6.1 Scalar Multiplication

The following easy exercise states that addition of a scalar constant to a function and multiplication of a function by a scalar both preserve measurability.

Exercise 3.37. Let (X, Σ) be a measurable space. Show that if $f \colon X \to \overline{\mathbb{R}}$ is measurable and $c \in \mathbb{R}$, then cf and $f + c$ are both measurable.

3.6.2 Addition

In order to consider measurability of a sum, we will need the following lemma, which relates two extended real-valued functions.

Lemma 3.38. *If $f,\ g \colon X \to \overline{\mathbb{R}}$ are measurable functions on a measurable space (X, Σ), then $\{f > g\}$ is a measurable subset of X.*

Proof. Since $\mathbb{Q}$ is countable, we can enumerate it, say as $\mathbb{Q} = \{r_k\}_{k\in\mathbb{N}}$. Then we can write $\{f > g\}$ as

$$\{f > g\} \;=\; \bigcup_{k=1}^{\infty} \{f > r_k > g\} \;=\; \bigcup_{k=1}^{\infty} \Big(\{f > r_k\} \,\cap\, \{g < r_k\}\Big).$$

Each set $\{f > r_k\}$ and $\{g < r_k\}$ is measurable because f and g are measurable functions, so $\{f > g\}$ is a measurable set. □

We need to be careful when we attempt to add extended real-valued functions, because we might encounter the indeterminate form $\infty - \infty$. If we restrict our attention to real-valued functions, we can use Lemma 3.38 to show that the sum of two real-valued measurable functions is measurable.

Theorem 3.39. *If $f,\ g \colon X \to \mathbb{R}$ are measurable functions on a measurable space (X, Σ), then $f + g$ is a measurable function.*

Proof. Given $a \in \mathbb{R}$, Exercise 3.37 implies that $a - g = a + (-1)g$ is measurable, so by applying Lemma 3.38 we see that the set

$$\{f + g > a\} \;=\; \{f > a - g\}$$

is measurable. Consequently $f + g$ is a measurable function. □

If f and g are extended real-valued functions, then the argument of Theorem 3.39 is not valid, because $f+g$ may not be defined at all points. Suppose that in addition to just having a measurable space (X,Σ), we have a measure μ on (X,Σ). If the measure μ is complete and f and g are measurable extended real-valued functions that are finite almost everywhere, then the indeterminate forms $f(x)+g(x)=\infty-\infty$ or $-\infty+\infty$ can at most occur on a set of measure zero. Hence $f+g$ is defined almost everywhere in the sense of Notation 3.29, and $f+g$ is measurable no matter what values we decide to assign it at points where it is undefined. We state this precisely as the next theorem.

Theorem 3.40. *Let (X,Σ,μ) be a complete measure space. If $f,\ g\colon X\to\overline{\mathbb{R}}$ are measurable functions that are finite μ-a.e., then $f+g$ is defined μ-a.e. on X and is measurable.* $\diamondsuit$

The next exercise addresses the more general setting where we either do not have a measure at all or do not know that f and g are finite a.e. This exercise shows that if we choose a single fixed constant and redefine $f(x)+g(x)$ to be that constant whenever it takes an indeterminate form, then $f+g$ will be measurable.

Exercise 3.41. Let $f,\ g\colon X\to\overline{\mathbb{R}}$ be measurable functions on a measurable space (X,Σ). Fix $c\in\overline{\mathbb{R}}$, and show that the function

$$h(x) \;=\; \begin{cases} c, & \text{if } f(x)=\infty \text{ and } g(x)=-\infty,\\ c, & \text{if } f(x)=-\infty \text{ and } g(x)=\infty,\\ f(x)+g(x), & \text{otherwise,} \end{cases}$$

is measurable on X. $\diamondsuit$

3.6.3 Products and Quotients

We use a cute trick to show that products of real-valued measurable functions are measurable.

Theorem 3.42. *If $f,\ g\colon X\to\mathbb{R}$ are measurable functions on a measurable space (X,Σ), then fg is measurable.*

Proof. The functions $f+g$ and $f-g$ are measurable by Theorem 3.39, so Corollary 3.34 implies that $(f+g)^2$ and $(f-g)^2$ are measurable. Applying Theorem 3.39 again, we see that

$$fg \;=\; \frac{(f+g)^2-(f-g)^2}{4}$$

is measurable. $\square$

Although a different proof technique is required, the next exercise states that Theorem 3.42 extends without change to extended real-valued functions. We do not even need to assume that f, g are finite almost everywhere.

Exercise 3.43. Given measurable functions $f,\, g\colon X \to \overline{\mathbb{R}}$ on a measurable space (X, Σ), show that fg is measurable. $\diamondsuit$

When dealing with quotients, we do need to avoid the indeterminate form $1/0$. According to the next exercise states, quotients of measurable functions are measurable as long as we avoid division by zero.

Exercise 3.44. Let $f,\, g\colon X \to \overline{\mathbb{R}}$ be measurable functions on a measurable space (X, Σ).

(a) Show that if $g(x) \neq 0$ for every $x \in X$, then f/g is measurable.

(b) Assume that μ is a complete measure on (X, Σ). Show that if $g \neq 0$ μ-a.e., then f/g is measurable. $\diamondsuit$

3.6.4 Complex-Valued Functions

In some ways, complex-valued functions are simpler than extended real-valued functions because there is no complex infinity. If f maps X into $\mathbb{C}$, then $f(x)$ is a complex number for each $x \in X$, and therefore the real and imaginary parts of $f(x)$ are finite real numbers.

The next exercise summarizes the interaction between arithmetic and measurability for complex-valued functions.

Exercise 3.45. Let $f,\, g\colon X \to \mathbb{C}$ be measurable functions on a measurable space (X, Σ), and prove the following statements.

(a) cf and $f + c$ are measurable for each $c \in \mathbb{C}$.

(b) $f + g$ is measurable.

(c) fg is measurable.

(d) If $g(x) \neq 0$ for every x, then f/g is measurable.

Also make the appropriate modification to part (d) if μ is a complete measure on (X, Σ) and g is nonzero μ-a.e. $\diamondsuit$

Let $\mathcal{F}_{\mathrm{m}}(X)$ (m for "measurable") denote the space of all complex-valued measurable functions on X, i.e.,

$$\mathcal{F}_{\mathrm{m}}(X) \;=\; \bigl\{f\colon X \to \mathbb{C} \;:\; f \text{ is measurable}\bigr\}.$$

It follows from parts (a) and (b) of Exercise 3.45 that $\mathcal{F}_{\mathrm{m}}(X)$ is a vector space over the complex field, and part (c) of that exercise also tells us that $\mathcal{F}_{\mathrm{m}}(X)$ is closed with respect to pointwise products of functions.

Additional Problems

3.12. Suppose that $f\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ and $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ are each measurable functions. Show that the function $F\colon \mathbb{R}^{m+n} \to \overline{\mathbb{R}}$ given by $F(x,y) = f(x)g(y)$ is measurable.