3.7 Suprema, Infima, and Limits

Now we turn to max, min, sup, inf, limsup, liminf, and limits. Of course, max and min are special cases of sup and inf, but it is nice to observe that we can deduce the measurability of the max and min of two real-valued functions by writing

$$\max\{f,g\} \;=\; \frac{f+g}{2} + \frac{|f-g|}{2}, \qquad \min\{f,g\} \;=\; \frac{f+g}{2} - \frac{|f-g|}{2}.$$

The next theorem gives a more general result for suprema and infima that also includes the case of extended real-valued functions.

Theorem 3.46. *If (X,Σ) is a measurable space and $f_k\colon X \to \overline{\mathbb{R}}$ is measurable for each $k \in \mathbb{N}$, then*

$$g(x) \;=\; \sup_k f_k(x) \qquad \text{and} \qquad h(x) \;=\; \inf_k f_k(x)$$

are measurable functions on X.

Proof. The proof follows by noting that

$$\Bigl\{\sup_k f_k > a\Bigr\} \;=\; \bigcup_{k=1}^{\infty} \{f_k > a\}$$

and

$$\Bigl\{\inf_k f_k > a\Bigr\} \;=\; \bigcup_{k=1}^{\infty} \{f_k < a\}. \qquad \diamondsuit$$

Since a limsup can be written as an "inf sup," we see that measurability is preserved with respect to limsups and liminfs.

Corollary 3.47. *If (X,Σ) is a measurable space and $f_k\colon X \to \overline{\mathbb{R}}$ is measurable for each $k \in \mathbb{N}$, then*

$$g(x) \;=\; \limsup_{k\to\infty} f_k(x) \qquad \text{and} \qquad h(x) \;=\; \liminf_{k\to\infty} f_k(x)$$

are measurable functions on X.

Proof. We just have to write

$$\limsup_{k\to\infty} f_k(x) \;=\; \inf_{k\in\mathbb{N}} \, \sup_{n\ge k} f_n(x)$$

and apply Theorem 3.46 twice. A similar argument applies to liminf. $\square$

Since $\lim f_k(x)$ exists if and only if $\limsup f_k(x) = \liminf f_k(x)$, we also have a result for limits. In the case where we have a complete measure on X, we only need to assume that the limit exists almost everywhere.

Corollary 3.48. *Let (X, Σ) be a measurable space and let $f_k \colon X \to \overline{\mathbb{R}}$ be measurable for each $k \in \mathbb{N}$.*

(a) *If $g(x) = \lim_{k\to\infty} f_k(x)$ exists for all $x \in X$, then g is measurable.*

(b) *Assume that μ is a complete measure on (X, Σ). If $g(x) = \lim_{k\to\infty} f_k(x)$ exists for μ-a.e. $x \in X$, then g is measurable.* $\diamondsuit$

By breaking into real and imaginary parts, we see that Corollary 3.48 carries over without any changes to measurable complex-valued functions $f_k \colon X \to \mathbb{C}$.

Additional Problems

3.13. Let (X, Σ) be a measurable space and suppose that $f_k \colon X \to \overline{\mathbb{R}}$ are such that $g(x) = \lim_{k\to\infty} f_k(x)$ exists for each $x \in X$. Given $a \in \mathbb{R}$, show that

$$\{g > a\} \;=\; \bigcup_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \bigl\{f_k \ge a + \tfrac{1}{n}\bigr\}.$$

Use this to give another proof that if each f_k is measurable, then g is also measurable.

3.14. Let (X, Σ) be a measurable space and let $f_k \colon X \to \overline{\mathbb{R}}$ be measurable for each $k \in \mathbb{N}$. Show that

$$\Bigl\{x \in X : \lim_{k\to\infty} f_k(x) \text{ exists}\Bigr\}$$

is a measurable subset of X.

3.15. (a) Let $f \colon \mathbb{R} \to \mathbb{R}$ be given. Show that if there exists a continuous function $g \colon \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e., then f is Lebesgue measurable.

(b) We say that $f \colon \mathbb{R} \to \mathbb{R}$ is continuous a.e. if it is continuous at almost every point, i.e., if

$$\lim_{y\to x} f(y) \;=\; f(x) \quad \text{for almost every } x \in \mathbb{R}.$$

Show that if f is continuous a.e. then it is Lebesgue measurable.

(c) Give an example of a function f that is continuous a.e., but such that there is no continuous function g that satisfies $f = g$ a.e.

3.16. Show that if $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable at almost every point, then f' is Lebesgue measurable.

3.17. A function $f \colon \mathbb{R}^d \to \overline{\mathbb{R}}$ is *upper semicontinuous* (abbreviated usc) at a point $x \in \mathbb{R}^d$ if

$$\limsup_{y\to x} f(y) \;\le\; f(x). \tag{3.6}$$

Precisely, equation (3.6) means that

$$\forall\,\varepsilon > 0, \quad \exists\,\delta > 0 \quad \text{such that} \quad |x - y| < \delta \implies f(y) \le f(x) + \varepsilon.$$

An analogous definition is made for *lower semicontinuity* (lsc). Prove the following facts.

(a) f is usc at every point $x \in \mathbb{R}^d$ if and only if $\{f \ge a\}$ is closed for each $a \in \mathbb{R}$.

(b) If we fix $r > 0$ and define

$$h(x) \;=\; \inf\bigl\{f(y) : y \in B_r(x)\bigr\},$$

where $B_r(x)$ is the open ball of radius r centered at x, then h is usc at every point. Is the same true if we replace $B_r(x)$ by the *closed* ball of radius r centered at x?

(c) Let K be a compact subset of $\mathbb{R}^d$. If f is finite at all points and f is usc at every point of K, then f is bounded above on K.