3.9 Egoroff's Theorem

We know that pointwise convergence of functions does not imply uniform convergence, and likewise pointwise a.e. convergence does not imply L^∞ norm convergence. A standard counterexample is the Shrinking Boxes of Example 3.62: $\chi_{[0,\frac{1}{k}]}$ converges pointwise a.e. to the zero function, but the convergence is not uniform. The functions $\chi_{[0,\frac{1}{k}]}$ are not continuous, but this is not the issue. For example,

$$f_k(x) = \begin{cases} 0, & x \le 0, \\ \text{linear}, & 0 < x < \frac{1}{2k}, \\ 1, & x = \frac{1}{2k}, \\ \text{linear}, & \frac{1}{2k} < x < \frac{1}{k}, \\ 0, & x \ge \frac{1}{k}. \end{cases}$$

is a continuous function and $f_k \to 0$ pointwise, but f_k does not converge uniformly to the zero function (see the illustration in Figure 3.1).

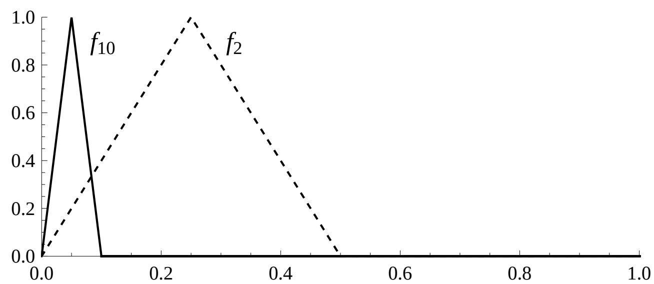

Fig. 3.1. Graphs of the functions f_2 (dashed) and f_{10} (solid).

However, if we allow ourselves to reduce the domain of these functions, then we can find a subset on which we have uniform convergence. In particular, if $0 < \delta < 1$ then the functions f_k converge uniformly to 0 on the interval $[\delta, 1]$. Egoroff's Theorem essentially states that this example is typical, as long as we are dealing with a finite measure space. So it is important in the example above that the region of interest (the interval $[0,1]$) has finite measure, but according to Egoroff, whenever we have pointwise a.e. convergence of a sequence of functions in a finite measure space, the sequence will actually converge uniformly on a "large" part of the domain.

Theorem 3.68 (Egoroff's Theorem). *Let (X, Σ, μ) be a finite measure space. If $f_k,\ f \colon X \to \mathbb{C}$ are measurable functions such that $f_k \to f$ pointwise a.e., then for every $\varepsilon > 0$ there exists a measurable set $E \subseteq X$ such that*

(a) $\mu(E) < \varepsilon$, *and*

(b) f_k *converges uniformly to* f *on* E^{C}, *i.e.,*

$$\lim_{k\to\infty} \Bigl(\sup_{x\notin E} |f(x) - f_k(x)|\Bigr) \;=\; 0.$$

Proof. Let Z be the set of measure zero consisting of all points $x \in X$ such that $f_k(x)$ does not converge to $f(x)$. For each $k, n \in \mathbb{N}$, define the measurable sets

$$E_k(n) \;=\; \bigcup_{m=k}^{\infty} \Bigl\{|f - f_m| \ge \frac{1}{n}\Bigr\} \qquad \text{and} \qquad Z_n \;=\; \bigcap_{k=1}^{\infty} E_k(n).$$

Fix n, and suppose that $x \in Z_n$. Then $x \in E_k(n)$ for every k, so for each k there must exist some integer $m \ge k$ such that $|f(x) - f_m(x)| > \frac{1}{n}$. Therefore $f_k(x)$ does not converge to $f(x)$ as k increases, so this point x belongs to Z. This shows that

$$Z_n \;\subseteq\; Z,$$

and therefore $\mu(Z_n) = 0$ by monotonicity. With n fixed, the sets $E_k(n)$ are nested decreasing, and their intersection is Z_n by definition. Therefore, it follows from continuity from above that

$$\forall\, n \in \mathbb{N}, \quad \lim_{k\to\infty} \mu\bigl(E_k(n)\bigr) \;=\; \mu(Z_n) \;=\; 0. \tag{3.13}$$

Fix $\varepsilon > 0$. Applying equation (3.13), for each integer n there is some integer $k_n > 0$ such that

$$\mu\bigl(E_{k_n}(k)\bigr) \;<\; \frac{\varepsilon}{2^n}.$$

Define

$$E \;=\; \bigcup_{n=1}^{\infty} E_{k_n}(n).$$

Subadditivity implies that $\mu(E) \le \varepsilon$. Further, if $x \notin E$ then $x \notin E_{k_n}(k)$ for any n, so $|f(x) - f_m(x)| < \frac{1}{n}$ for all $m \ge k_n$.

In summary, $\mu(E) \le \varepsilon$ and for each $n \in \mathbb{N}$ there exists an integer $k_n > 0$ such that

$$m \;\ge\; k_n \quad \Longrightarrow \quad \sup_{x\notin E} |f(x) - f_m(x)| \;\le\; \frac{1}{n}.$$

This says that f_k converges uniformly to f on E^{C}. $\square$

Additional Problems

3.23. Let (X, Σ, μ) be a finite measure space, and let $f_k,\, f\colon X \to \mathbb{R}$ be measurable function on X. Show that if $f_k \to f$ pointwise a.e., then $f_k \overset{\mathrm{m}}{\to} f$.

3.24. Let D be a Lebesgue measurable subset of $\mathbb{R}^d$ such that $|D| < \infty$. Show that if $f_k,\ f\colon D \to \mathbb{C}$ are measurable and $f_k \to f$ a.e. on D, then there exists a closed set $F \subseteq D$ such that $|D \backslash F| < \varepsilon$ and $f_k \to f$ uniformly on F.

3.25. Suppose that μ is a σ-finite measure on X, and $f_k,\ f\colon X \to \mathbb{C}$ are measurable functions such that $f_k \to f$ a.e. Show that there exist measurable sets $E_n \subseteq E$ such that

(a) $\mu\big(X \setminus \cup E_n\big) = 0$, and

(b) $f_k \to f$ uniformly on each set E_n.