The following is a simple, but extremely useful inequality.

Tchebyshev's Inequality

If $f \in L^1(X)$, then

$$\forall \varepsilon > 0, \quad \mu\{|f| \geq \varepsilon\} \leq \frac{1}{\varepsilon} \|f\|_1 = \frac{1}{\varepsilon} \int |f|.$$

Proof:

$$\begin{aligned} \|f\|_1 &= \int |f| \\ &\geq \int_{\{|f| \geq \varepsilon\}} |f| \\ &\geq \int_{\{|f| \geq \varepsilon\}} \varepsilon \\ &= \varepsilon \mu\{|f| \geq \varepsilon\}. \quad \blacksquare \end{aligned}$$

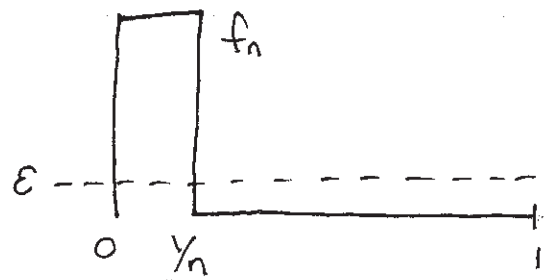
Exercise

Use Tchebyshev's Inequality to prove that

$$f_n \to f \text{ in } L^1(X) \implies f_n \xrightarrow{m} f.$$

One of our standard examples shows that the converse fails.

<u>Example</u>: Shrinking Boxes

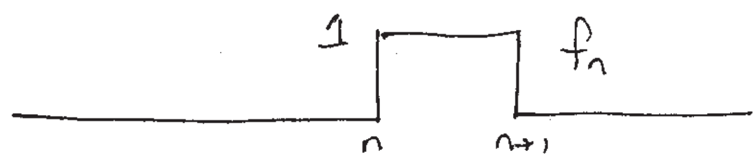


$\mu\{|f_n - 0| \geq \varepsilon\} = \frac{1}{n} \to 0$ (Lebesgue measure)

so $f_n \xrightarrow{m} 0$. However, $\|f_n - 0\|_1 = 1 \ \forall n$,

so $f_n \not\to 0$ in $L^1[0,1]$.

<u>Exercise:</u> Marching to infinity



Show $f_n \to 0$ pointwise everywhere,

$f_n \overset{m}{\not\to} 0$

$f_n \not\to 0$ in $L^1(\mathbb{R})$

What about the "circular marching boxes" example in $[0,1]$?