$L^p[0,1]$ for $0<p<1$

Definition

Let X be a vector space.

The convex hull of $E \subseteq X$ is the smallest convex set that contains E, i.e.,

$$\operatorname{co} E = \bigcap \{ K \subseteq X : K \text{ convex } \& \ K \supseteq E \}.$$

If X is a TVS then the closed convex hull $\overline{\operatorname{co}}\, E$ is the smallest closed convex set that contains E.

Theorem

If $0<p<1$ then $L^p[0,1]$ contains no nontrivial open convex subsets. Consequently $L^p[0,1]$ is not locally convex.

Proof:

Suppose that $K \subseteq L^p[0,1]$ is open & convex, $K \neq \emptyset$.

Translate K so that $0 \in K$ (Exercise: a translation of an open convex set in $L^p[0,1]$ is still open & convex).

Since K is open, $\exists\, r>0$ s.t.

$$B_r(0) \subseteq K.$$

Choose $f \in B_r(0)$. Then

$$s = \|f\|_p^p = d(f,0) < r.$$

Since $|f|^p \in L^1[0,1]$, the function

$$F(t) = \int_0^t |f(x)|^p \, dx$$

is a continuous, increasing function of t. Further,

$$F(0) = 0 \qquad \& \qquad F(1) = \int_0^1 |f(x)|^p\,dx = s.$$

Therefore, $\exists\, t \in [0,1]$ such that

$$\int_0^t |f(x)|^p\,dx = \frac{s}{2}.$$

Set

$$g = f \cdot \chi_{[0,t]} \qquad \& \qquad h = f \cdot \chi_{[t,1]}$$

Then

$$f = g + h = \tfrac{1}{2} \cdot 2g + \tfrac{1}{2} \cdot 2h.$$

We have

since $p < 1$

$$\|2g\|_p^p = \int_0^t |2f(x)|^p\,dx$$

$$= 2^p \cdot \frac{s}{2} = 2^{p-1} s < 2^{p-1} r < r$$

and

$$\|2h\|_p^p = \int_t^1 |2f(x)|^p\,dx$$

$$= 2^p \cdot \frac{s}{2} = 2^{p-1} s < 2^{p-1} r < r$$

Hence

$$2g, 2h \in B_{2^{p-1}r}(0).$$

But f lies on the line segment joining $2g$ to $2h$,

so

$$f \in co\, B_{2^{p-1}r}(0).$$

This is true for every $f \in B_r(0)$, so

$$B_r(0) \subseteq co\, B_{2^{p-1}}(0)$$

Rescaling,

$$B_{2^{1-p}r}(0) \subseteq co\, B_r(0).$$

Repeat the entire argument with $2^{1-p}r$ in place of r; we get

$$B_{2^{1-p}\cdot 2^{1-p}r}(0) \subseteq co\, B_{2^{1-p}r} \subseteq co\, B_r(0)$$

By induction,

$$B_{2^{n(1-p)}r}(0) \subseteq co\, B_r(0) \subseteq K.$$

since K is convex

Hence

$$L^p[0,1] = \bigcup_{n=1}^{\infty} B_{2^{n(1-p)}r} \subseteq K.$$

Hence $K = L^p[0,1]$. ∎