MATH 6338 (C. Heil) MIDTERM EXAM June 16, 2008

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (10 points) any other TWO problems (15 points each). You may also answer (for up to 5 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem. There are problems on BOTH SIDES of this page!

All statements require proof or justification. There are 40 points total, plus up to 5 points of extra credit.

Definition. Let X be a Banach space, and let f_n, f be vectors in X. Then we say that f_n *converges weakly to* f, denoted $f_n \xrightarrow{\text{w}} f$, if

$$\forall\, \mu \in X^*, \quad \lim_{n\to\infty} \langle f_n, \mu\rangle \;=\; \langle f, \mu\rangle.$$

1. (a) Let X be a Banach space, and f_n, $f \in X$. Show that if $f_n \to f$ (convergence in the norm of X), then $f_n \xrightarrow{\text{w}} f$.

(b) Now let $X = H$ be a Hilbert space, and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in H. Prove that $e_n \xrightarrow{\text{w}} 0$, but $e_n \not\to 0$ (i.e., we do not have convergence in the norm of H).

2. Let H be a Hilbert space. The two parts of this problem are not related.

(a) Show that if A is any subset of H, then $A^\perp$ is a closed subspace of H, and $(A^\perp)^\perp = \overline{\text{span}}(A)$.

(b) Let M be a closed subspace of H. Show that if $f \in H \setminus M$, then there exists $\mu \in H^*$ such that $\langle f, \mu\rangle = 1$ and $\langle h, \mu\rangle = 0$ for every $h \in M$.

3. Prove directly that $(c_0)^* \cong \ell^1$, i.e., $(c_0)^*$ and ℓ^1 are isometrically isomorphic.

4. Let $f \in L^1(\mathbb{R})$ and $g \in C_b^1(\mathbb{R})$ be given (i.e., g is differentiable everywhere, and g, g' are both continuous and bounded). Define their convolution by

$$(f * g)(x) \;=\; \int f(y)\, g(x-y)\, dy.$$

Show that $f * g$ is differentiable everywhere, and both $f * g$ and $(f * g)'$ are bounded.